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A Note on Applications of the Mild-slope Equation for Random Waves

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ABSTRACT

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Based on the formal derivation of the mild-slope equation (Smith & Sprinks, 1975), the neglected 'forcing terms' are rederived. It is shown that the slope terms are of order ϵ^2, which are negligible according to the mild-slope assumption, where $\epsilon = |\nabla h|/kh$ represents the classical definition of small parameter for mild-slope. It is found that the curvature terms depend not only on ϵ but also on the wave frequency and the terms have considerable effect on wave phase in the lower range of wave frequencies. Therefore, in the broad wave spectrum case, the curvature terms discarded by Smith & Sprinks (1975) are necessary to accurately predict waves over a bottom with severe curvatures.

ADDITIONAL INDEX WORDS: *Green's identity method, modified mild-slope equation, finite element solutions, wave dispersion relation, Bragg resonance scattering.*

INTRODUCTION

For last two decades, the mild-slope equation (MSE) has received extensive investigations since it was derived by BERKHOFF (1972) and SMITH and SPRINKS (1975). After VINCENT and BRIGGS (1989) undertook an experiment of the propagation of random waves over an elliptic shoal, many numerical models based on the MSE have been developed for random waves. PANCHANG *et al.* (1990) have found that the parabolic approximation method (PAM) may not be uniformly valid for all the frequencies in their numerical computation. They suggested that research efforts are needed to make the complete refraction-diffraction equation computationally viable for a larger range of frequencies. The PAM results from the omission of the curvature of the surface wave amplitude (*i.e.* A_{xx} in Eq.36) in the MSE. Detailed error analyses of both PAM and MSE are presented by SOBEY (1993). O'REILLY and GUZA (1991) compared the PAM with other models and found that in the broad wave spectrum case, using a single unidirectional wave to predict the wave field produced significant errors. Other models based on the MSE are also reported by KIRBY *et al.* (1992), and OZKAN and KIRBY (1993).

Using the variational principles and the Galerkin method, the modified MSE (MMSE) is given by CHAMBERLAIN and PORTER (1995). Due to the problems in random wave modeling, the derivation of the MSE based on Green's identify method is revisited. On investigation (EDGE and ZHANG, 1996), it is found that not all terms neglected in the derivation of the MSE are of $O(\epsilon^2)$ as justified by SMITH and SPRINKS (1975), where $\epsilon = |\nabla h|/kh$ represents the classical definition of a small parameter for a mild-slope. It is shown that slope terms are of $O(\epsilon)$ relative to the curvature terms, which are not small enough to be neglected in the low frequency-waves. In other words, the MSE approximations are not uniform for all frequencies of waves, and especially in the low frequency range, the errors induced by neglecting the additional curvature terms become larger. Therefore, the retention of additional curvature terms can reduce the errors of wave modeling based on the MSE.

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DERIVATION OF MSE BASED ON GREEN'S IDENTITY METHOD

For a small-amplitude wave with undisturbed angular frequency ω, it is assumed that the flow is incompressible and irrotational and that the pressure is constant at the free surface. The rectilinear coordinates (x,y,z) are fixed in space and $z = 0$ is located at the calm water level. The wave potential $\Phi(x,y,z)$ satisfies the equations:

$$\nabla^2\Phi + \Phi_{zz} = 0, \qquad (-h \le z \le 0), \tag{1}$$

$$\Phi_{tt} + g\Phi_z = 0, \qquad \text{at } z = 0, \tag{2}$$

Since the sea bottom is fixed at $z = -h(x, y)$, the normal velocity vanishes; this implies

$$\Phi_z + \nabla h \cdot \nabla\Phi = 0, \quad \text{at } z = -h, \tag{3}$$

where ∇ denotes the horizontal gradient operator, *i.e.* $\nabla = \{\partial/\partial x, \partial/\partial y\}$.

The depth-integrated wave equation for monochromatic, linear waves propagating over a uneven bottom may be formulated following the Green's formula method of SMITH and SPRINKS (1975). The solution to (1)–(3) may be expressed as

$$\Phi(x, y, z, t) = f(q, Q)\phi(x, y, t) + (\text{non-propagating modes}) \quad (4)$$

where

$$f = \cosh Q/\cosh q \quad (5)$$

is a function of z,k, and h. Throughout this paper the notations $q = kh$, $Q = \tanh k(z + h)$, and $\sigma = \tanh kh$ are used for convenience. The propagation of waves is associated only with the propagating mode, thus extracting this component and applying Green's identity to f and Φ:

$$\int_h^0 (f\Phi_{zz} - \Phi f_{zz})\, dz = [f\Phi_z - \Phi f_z]^0_h \quad (6)$$

or

$$\int_h^0 (f\nabla^2\Phi + \Phi f_{zz})\, dz = -[f\Phi_z - \Phi f_z]^0_h \quad (7)$$

Using Eqs. (4) and (5)

$$f_{zz} = k^2 f \quad (8)$$

$$\nabla\Phi = f\nabla\phi + \phi\nabla f \quad (9)$$

$$\nabla^2\Phi = f\nabla^2\phi + 2\nabla\phi\cdot\nabla f + \phi\nabla^2 f \quad (10)$$

$$\Phi_z|_{z=h} = -\nabla h\cdot\nabla\Phi = -\nabla h\cdot(f\nabla\phi + \phi\nabla f) \quad (11)$$

Inserting Eqs. (8)–(11) into Eq. (7)

$$\int_h^0 (\phi k^2 f^2 + \nabla^2\phi f^2 + 2f\nabla\phi\cdot\nabla f + \phi f\nabla^2 f)dz = -\nabla h\cdot(f\nabla\phi + \phi\nabla f)f|_{z=h} \quad (12)$$

Based on Eq. (5), the terms in Eq. (12) are evaluated using the following

$$\nabla f = f_h\nabla h + f_k\nabla k \quad (13)$$

$$\nabla^2 f = f_{hh}\nabla h\cdot\nabla h + f_h\nabla^2 h + 2f_{hk}\nabla h\cdot\nabla k + f_k\nabla^2 k + f_{kk}\nabla k\cdot\nabla k \quad (14)$$

$$f_h = k(\sinh Q - \sigma\cosh Q)/\cosh q \quad (15)$$

$$f_k = (Q\sinh Q - q\sigma\cosh Q)/(k\cosh q) \quad (16)$$

$$f_{hh} = 2\sigma k^2(\sigma\cosh Q - \sinh Q)/\cosh q \quad (17)$$

$$f_{kk} = \{Q^2\cosh Q - 2\sigma qQ\sinh Q - q^2(1 - 2\sigma^2)\cosh Q\}/(k^2\cosh q) \quad (18)$$

$$f_{hk} = \{(2q\sigma^2 - \sigma - q)\cosh Q + (1 - q\sigma)\sinh Q + Q\cosh Q - Q\sigma\sinh Q\}/\cosh q \quad (19)$$

where $f_h = \partial f/\partial h$, $fk = \partial f/\partial k$, $f_{hh} = \partial^2 f/\partial h^2$, $f_{hk} = \partial^2 f/\partial h\partial k$, $f_{kk} = \partial^2 f/\partial k^2$.

Using Leibniz's rule with Eq. (12),

$$\int_h^0 (\nabla^2\phi f^2 + 2f\nabla\phi\cdot\nabla f)dz + f^2\nabla h\cdot\nabla\phi|_{z=h} = \nabla\cdot(CC_g\nabla\phi)/g \quad (20)$$

Substituting Eqs. (13)–(20) into Eq. (12) and noting

$$g\int_h^0 k^2\phi f^2\, dz = k^2 CC_g\phi \quad (21)$$

$$2k\int_h^0 \sinh Q\cosh Q\, dz = \sigma^2/(1 - \sigma^2) \quad (22)$$

$$4k\int_h^0 Q\sinh Q\cosh Q\, dz = \{q(1 + \sigma^2) - \sigma\} / ((1 - \sigma^2)) \quad (23)$$

$$4k\int_h^0 Q\cosh Q\cosh Q\, dz = \{q^2((1 - \sigma^2)) + 2q\sigma - \sigma^2\}/((1 - \sigma^2)) \quad (24)$$

$$4k\int_h^0 Q^2\cosh Q\cosh Q\, dz = \{2q^3(1 - \sigma^2) + 2q^2\sigma - q(1 + \sigma^2) + \sigma\} / (1 - \sigma^2) \quad (25)$$

the modified MSE equation governing potential $\phi(x, y)$ and wavenumber $k(x, y)$ is obtained:

$$\phi_{tt} - \nabla\cdot(CC_g\nabla\phi) + (\omega^2 - k^2CC_g)\phi - gk[\alpha_1(\nabla h\cdot\nabla h) + \alpha_2\nabla^2 h/k + \alpha_3\nabla k\cdot\nabla h/k^2 + \alpha_4\nabla^2 k/k^3 + \alpha_5(\nabla k\cdot\nabla k)/k^4]\phi = 0 \quad (26)$$

where h is total water depth, C and C_g are the wave celerity (ω/k) and the group velocity before bottom disturbance, respectively. The dimensionless parameters of α_i ($i = 1, 5$) are

$$\alpha_1 = -\sigma(1 - \sigma^2)(1 - \sigma q) \quad (27)$$

$$\alpha_2 = -\sigma q(1 - \sigma^2)/2 \quad (28)$$

$$\alpha_3 = 2\sigma^2 q^2(1 - \sigma^2) - q(1 - \sigma^2)(5\sigma + q)/2 \quad (29)$$

$$\alpha_4 = q(1 - \sigma^2)(1 - 2\sigma q)/4 - \sigma/4 \quad (30)$$

$$\alpha_5 = q(1 - \sigma^2)(4\sigma^2 q^2 - 4q^2/3 - 2\sigma q - 1)/4 + \sigma/4 \quad (31)$$

All terms in the bracket of Eq. (26) are called 'forcing' terms by SMITH and SPRINKS (1975). However, they mistakenly omitted all terms related to f_k, f_{kk}, f_{hk} in the 'forcing terms'. This mistake is cited later by several authors, as indicated in MEI (1983, p87). Furthermore, assuming terms of $O(\epsilon^2)$, SMITH and SPRINKS (1975) neglected all the 'forcing terms'. In other words, by discarding the terms in the bracket of Eq. (26), the mild-slope equation is recovered from Eq. (26), *i.e.*

$$\Phi_{tt} - \nabla\cdot(CC_g\nabla\Phi) + (\omega^2 - k^2CC_g)\Phi = 0 \quad (32)$$

Note that the different approach from CHAMBERLAIN and PORTER'S (1995) is used to derive Eq.(26). Although CHAMBERLAIN and PORTER'S (1995) Eq. (2.12) is equivalent to Eq.(26) in steady state, if the leading-order dispersion relation Eq.(34) is invoked, Eq.(26) is more general regarding the total dispersion relation Eq.(33) in next section.

TOTAL DISPERSION RELATION AND REDUCED WAVE EQUATION

In this section, an analysis is given to show orders of the slope and curvature terms in Eq.(26). Assume that the velocity potential is given by $\Phi = Ae^{i\theta}$, where A and $\tilde{\theta}$ are real, then for monochromatic waves $\tilde{\kappa} = \nabla\tilde{\theta}$ and $\tilde{\omega} = -\partial\tilde{\Phi}/\partial t$. Substituting $\Phi = Ae^{i\Phi}$ into Eq.(26), the wave equation is split into real and imaginary parts. Similar to the derivation of LIU (1990), the real part leads to the wave-action conservation, while the imaginary part leads to the total dispersion relation:

$$\tilde{\omega}^2 = \omega^2 + CC_g(\tilde{k}^2 - k^2) + \frac{A_{tt}}{A} - CC_g\frac{\nabla^2 A}{A} - \frac{\nabla(CC_g)\cdot\nabla A}{A} - gk[\alpha_1(\nabla h\cdot\nabla h) + \alpha_2\nabla^2 h/k + \alpha_3\nabla k\cdot\nabla h/k^2 + \alpha_4\nabla^2 k/k^3 + \alpha_5(\nabla k\cdot\nabla k)/k^4], \quad (33)$$

where

$$\omega^2 = gk \tanh kh \quad (34)$$

is the leading-order dispersion relation.

If the frequency is fixed and temporal variation of amplitude is not considered, the effective wave number as defined by LIU (1990) is

$$\tilde{k}^2 = k^2 + \frac{\nabla^2 A}{A} + \frac{\nabla(CC_g)\cdot\nabla A}{CC_g A} + \frac{gk}{CC_g} \times [\alpha_1(\nabla h\cdot\nabla h) + \alpha_2\nabla^2 h/k + \alpha_3\nabla k\cdot\nabla h/k^2 + \alpha_4\nabla^2 k/k^3 + \alpha_5(\nabla k\cdot\nabla k)/k^4] \quad (35)$$

Discarding the terms in the bracket, the effective wave number for the MSE (LIU, 1990, p35)

$$\tilde{k}^2 = k^2 + \frac{\nabla^2 A}{A} + \frac{\nabla(CC_g)\cdot\nabla A}{CC_g A} \quad (36)$$

is recovered. Based on Eq.(36), the direction of wave propagation can be computed as indicated by GIROLAMO (1995) for irregular waves. In steady state, applying the phase conservation to the leading-order dispersion relation, *i.e.* $\nabla\omega = 0$, the slope and curvature of the wave number are

$$\frac{\nabla k}{k} = \beta_1\frac{\nabla h}{h}, \quad (37)$$

and

$$\frac{\nabla^2 k}{k} = \beta_1\frac{\nabla^2 h}{h} + \beta_2\frac{(\nabla h\cdot\nabla h)}{h^2} \quad (38)$$

respectively, where

$$\beta_1 = -q(1 - \sigma^2)/\gamma, \quad (39)$$

$$\beta_2 = 2q^2(1 - \sigma^2)(\gamma - \alpha_1)/\gamma^3, \quad (40)$$

$$\gamma = \sigma + q(1 - \sigma^2). \quad (41)$$

From wave-action conservation in steady state

$$\nabla\cdot(\vec{C}_g A^2) = 0, \quad (42)$$

the slope and curvature of amplitude are obtained as

$$\frac{\nabla A}{A} = \beta_3\frac{\nabla h}{h}, \quad (43)$$

and

$$\frac{\nabla^2 A}{A} = \beta_3\frac{\nabla^2 h}{h} + \beta_4\frac{(\nabla h\cdot\nabla h)}{h^2}, \quad (44)$$

repectively, where

$$\beta_3 = q\alpha_1/\gamma^2 \quad (45)$$

$$\beta_4 = q^2\{5\alpha_1^2 - 2\gamma\sigma^2\alpha_1 + \sigma^2\gamma^2\}/\gamma^4. \quad (46)$$

Using

$$CC_g = g\gamma/(2k), \quad (47)$$

and Eq. (37), it is noted that

$$\frac{\nabla(CC_g)}{CC_g} = \beta_5\frac{\nabla h}{h}, \quad (48)$$

where

$$\beta_5 = q(1 - \sigma^2)\{2\sigma(1 - q\sigma) + \gamma\}/\gamma^2 \quad (49)$$

From Eq. (36), the wavenumber square difference for the MSE is obtained

$$\tilde{k}^2 - k^2 = \mu_1\frac{\nabla^2 h}{h} + \nu_1\frac{(\nabla h\cdot\nabla h)}{h^2} \quad (50)$$

where

$$\mu_1 = \beta_3 \quad (51)$$

$$\nu_1 = \beta_4 + \beta_3\beta_5. \quad (52)$$

Similarly, from Eq.(35), the wavenumber square difference for the modified MSE

$$\tilde{k}^2 - k^2 = \mu_2\frac{\nabla^2 h}{h} + \nu_2\frac{(\nabla h\cdot\nabla h)}{h^2} \quad (53)$$

where

$$\mu_2 = \mu_1 + 2q(\alpha_2 + \alpha_4\beta_1/q)/\gamma \quad (54)$$

$$\nu_2 = \nu_1 + 2q^2(\alpha_1 + \alpha_3\beta_1/q + \alpha_4\beta_2/q^2 + \alpha_5\beta_1^2/q^2)/\gamma. \quad (55)$$

Since the additional curvature terms ($\mu_2 - \mu_1$) and slope terms ($\nu_2 - \nu_1$) are beyond the prediction of the MSE, the numerical 'breaking down' on ripple beds occurs (Figure 2), which was first shown by KIRBY (1986).

First, for simplicity, two dimensional waves over sinusoidal beds are treated here, the variation of wavenumber square due to bottom undulation is obtained from Eq.(53)

$$\tilde{k}^2 - k^2 = \mu_2\frac{h_{xx}}{h} + \nu_2\frac{(h_x)^2}{h^2}. \quad (56)$$

Assuming bed form

$$h(x) = h_0 - b\sin Kx \quad (57)$$

where h_0 is the mean water depth, K is the bottom undulation wave number, and b is the bed amplitude,

$$(h_x)^2 = (Kb)^2\cos^2 Kx \quad (58)$$

$$h_{xx} = K^2 b\sin Kx \quad (59)$$

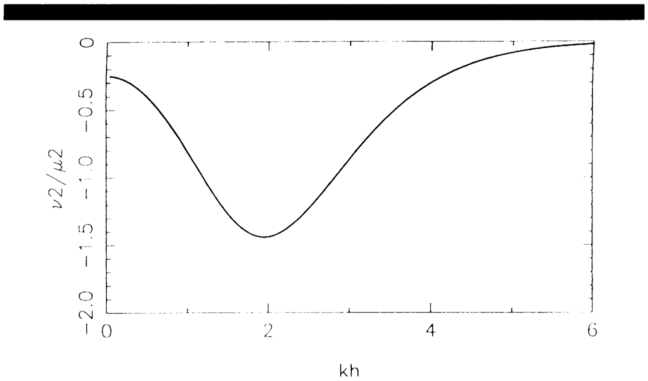

Figure 1. Ratio of Slope to Curvature Function.

Thus the relative wavenumber square difference for the modified MSE is

$$\frac{\tilde{k}^2 - k^2}{k^2} = \mu_2 \frac{K}{k}(\epsilon)\sin Kx + \nu_2\epsilon^2\cos^2 Kx, \tag{60}$$

where $\epsilon = |\nabla h|/kh$. In Eq.(60) the first term represents the curvature effects and the second term represents the slope effects on the wave phase. By inspection of Eq.(60), it is found that curvature effects depend not only on the bottom slope (ϵ) but also the ratio K/k. In other words, the curvature effects depend on both bottom slope and surface wave frequency if the topography is fixed.

The orders of both slope and curvature are studied by examining the ratio of slope to curvature terms:

$$R = \frac{\nu_2}{\mu_2}\frac{k}{K}\frac{Kb}{kh}\cos Kx/\tanh Kx \tag{61}$$

Since the range of ν_2/μ_2 is about $0 \sim -1.5$ (see Figure 1), Eq.(61) shows that, except $k/K\epsilon = O(1)$, bottom slope and curvature terms are not of same order.

Up to this stage, it is clear that the additional slope terms are of $O(\epsilon)$ relative to the curvature terms. In addition to bottom slope, the curvature effects are also dependent on the ratio of bottom wave number to surface wave number, or in other words, the wave frequency for a fixed bottom. Therefore, being consistent with the assumption of the MSE, the additional slope terms are neglected and Eq.(26) is reduced to

$$\phi_{tt} - \nabla\cdot(CC_g\nabla\phi) + (\omega^2 - k^2CC_g)\phi + gF\phi\nabla^2h = 0 \tag{62}$$

where

$$F = \frac{(1 - \sigma^2)[q(1 + \sigma^2) - \sigma]}{4\gamma} \tag{63}$$

In Eq.(62), the last term is the additional curvature term if compared with the MSE. For small bottom slopes, the slope terms are of $O(\epsilon^2)$. Therefore, the slope terms in the 'forcing terms' can be neglected according to the mild-slope assumption. Eq.(62) is referred to as reduced modified mild-slope equation (RMMSE).

It is understood that the curvature effects are determined by two factors: bottom slope and surface wave frequency (or ratio of bottom wave number to surface wave number). In the broad-spectrum random waves, this ratio has large ranges and the resulting accuracy of the MSE solution is quite different from one wave component to another. If this ratio, K/k, is not of the order of the bottom slope, the approximation of the MSE may not be uniform. This is demonstrated, for two dimensional waves, by comparing numerical solutions with experimental data in the next Section.

NUMERICAL EXAMPLES AND DISCUSSIONS

For the monochromatic and steady waves, the one-dimensional MMSE Eq.(26) and the one dimensional RMMSE Eq.(62) become

$$(CC_g\phi_x)_x + k^2CC_g\phi + gk[\alpha_1(h_x)^2 + \alpha_2h_{xx}/k + \alpha_3h_xk_x/k^2 + \alpha_4k_{xx}/k^3 + \alpha_5(k_x)^2/k^4]\phi = 0, \tag{64}$$

and

$$(CC_g\phi_x)_x + k^2CC_g\phi - gF\phi h_{xx} = 0, \tag{65}$$

respectively. Both Eq.(64) and Eq.(65) are solved using higher order finite element method. The numerical schemes of the MSE, the MMSE and RMMSE are presented in ZHANG and EDGE (1996).

If waves pass over a sinusoidal bottom with a matching of phases between the surface wave and bottom undulation, strong reflection of waves occur at certain frequencies. This is the well-known Bragg resonance phenomena and can be used as an indicator of wave modeling accuracy. As shown above, the major effects of the discarded terms are on wave phase, so computations of wave reflection by sinusoidal beds are suitable examples to test the accuracy of the MSE, the MMSE and the RMMSE by comparing with experimental data from DAVIES and HEATHERSHAW (1984).

In Figure 2, as first shown by KIRBY (1986), the MSE failed to predict the Bragg resonance peak. Here it is emphasized that due to the discarded curvature terms, the major errors of the MSE are within low frequency ranges for a single sinusoidal bed, and both the MMSE and RMMSE perform well if compared with experimental data. Figure 3 indicates that as the water depth becomes shallow, the difference among the MSE, MMSE and RMMSE becomes smaller, but the MSE still presents some errors. Inspection of Figure 4 shows that the MSE tends to overestimate the reflection coefficients at relatively higher frequencies and to under-predict at low frequencies for doubly-sinusoidal beds. An additional example for doubly-sinusoidal beds is presented in Figure 5. The first peak is due to the first-harmonic associated with the longer bed wavelength and the second-harmonic resonance associated with the shorter bed wavelength. The MSE under-predicts the former and over-predicts the latter so the overall results happen to be closer to real solutions. However, for the second peak the results of the MSE are far from the experimental data of GUAZZELLI *et al.* (1992).

One may argue that the parameter ($\epsilon = |\nabla h|/kh$) should be small enough for the MSE to be valid for the computed examples. The typical bed slope here is 0.314, thus the slope terms are of the order of 0.09 in Eq.(60). Since the only difference between the MSE and the RMMSE is additional cur-

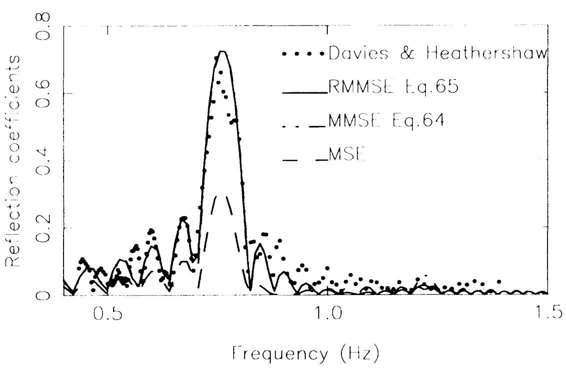

Figure 2. Sinusoidal bed, n = 10, b/h0 = 0.16.

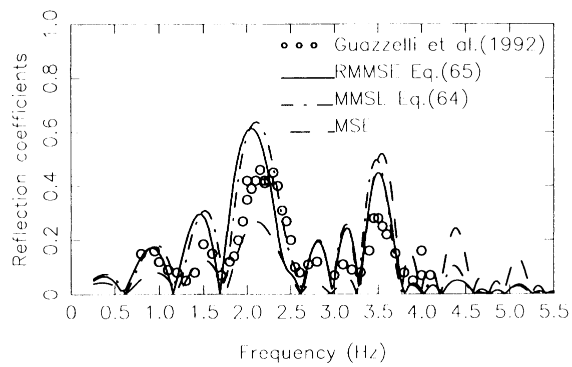

Figure 4. Doubly−sinusoidal bed, n = 4,m = 2,b/h0 = 0.25.

vature terms, which are frequency-dependent, even for very small bed slopes, the curvature effects may not be very small for some wave components. The most important factor is that the slope terms are $O(\epsilon)$ relative to the curvature terms. Therefore, the numerical results of the MMSE and the RMMSE have little difference for single sinusoidal bed (Figure 2 and Figure 3). For doubly-sinusoidal bed cases, the difference between the MMSE and RMMSE becomes larger (Figures 4 & 5), because the second sinusoidal component is steeper than the first one. It is clearly understood from Eq.(60) that for small slope bottoms, the RMMSE is able to model waves as accurate as the MMSE. Other examples of comparison between the RMMSE and the MMSE and the applications of the RMMSE in random waves are presented in EDGE and ZHANG (1996).

For waves over shoals with strong curvature, VINCENT and BRIGGS (1989) found that the regular waves cannot provide a good representation of the irregular waves in most cases, especially for the directionally spreading wave case. This may be due partially to the frequency dependent nature of the curvature terms. By recasting the MSE into a system of so--called energy transport and eikonal equations, the direction of wave propagation can be computed as suggested by GIROLAMO (1995) for irregular waves. Comparing Eq.(35) and (36), it is clear that the additional curvature terms are of same order as the original terms of the MSE. Therefore, the incomplete curvature terms in the eikonal equation (36) of the MSE may induce some errors, and including the additional curvature terms in the eikonal equation (35) of the reduced wave equation can reduce the errors. It is also illustrated numerically that the additional curvature terms beyond the MSE have significant effects on wave reflection coefficients in low frequency regions in Figures 2–5. As to three-dimensional wave transformation, these additional curvature terms may cause the wave to change direction. Also due to the frequency-dependent nature of the additional curvature terms, the MSE may produce errors for modeling irregular directionally spreading waves.

CONCLUSION

After a theoretical and numerical analysis, the RMMSE (Eq. 62) is proposed to predict random wave transformation over mild topography. It is found that the additional curvature effects strongly depend on the wave frequency and the MSE may induce some errors in predicting the wave reflection if the bottom undulation is more rapid than the surface waves, *i.e.* in the lower range of wave frequencies for random waves. The solutions of the RMMSE are shown to be in better agreement with laboratory data than those of the MSE.

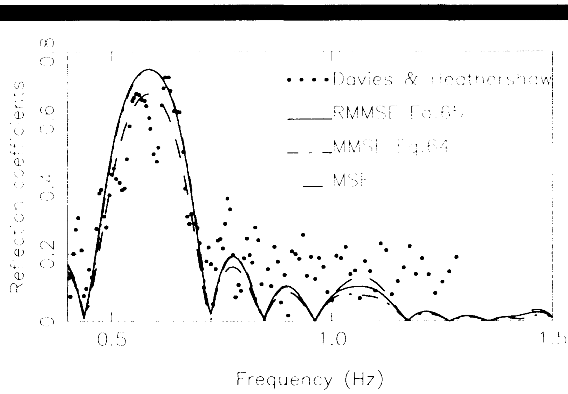

Figure 3. Sinusoidal bed, n = 4, b/h0 = 0.32.

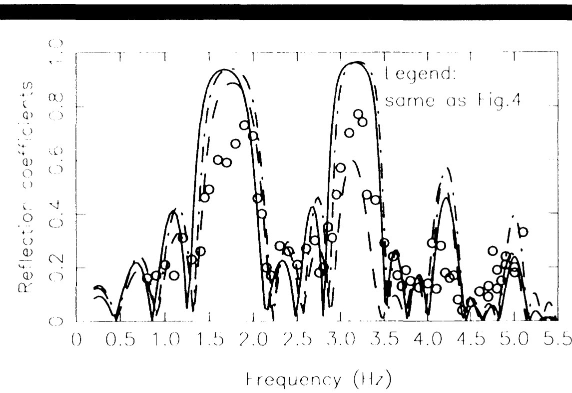

Figure 5. Doubly−sinusoidal bed, n = 4,m = 2,b/h0 = 0.4.

Therefore, it is concluded that the recommended RMMSE can improve the prediction of random waves in the lower range of wave frequencies.

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