A Unified Model for Periodic Non-Linear Dispersive Waves in Intermediate and Shallow Water

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ABSTRACT

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A numerical model for non-linear dispersive monochromatic wave propagation is developed in this work. The new model has a unified form, being valid in shallow as well as in intermediate water. The approach is based on the expansion of the vertical velocity in power series and on an analytical solution of the Laplace equation. It has a similar form with two types of the Boussinesq equations but instead of the constant coefficient 1/3 (or 1/15) in the momentum equation it is proposed a function of the water depth and the wave period. The continuity equation, which is exact in deep, intermediate and shallow water without any restriction in nonlinearity, remains unchanged. In the momentum equation terms of order up to $O(\epsilon\sigma^2)$—with ϵ=H/d, σ=d/L (H=wave height, d=water depth, L=wave length)—are considered. The horizontal and the vertical velocity as well as the pressure distribution are given in relation to the wave period and the instantaneous depth averaged horizontal velocity. The model is validated both in intermediate and shallow water against the non-linear theory and experimental data.

ADDITIONAL INDEX WORDS: *Numerical model, wave theory, Boussinesq equations.*

INTRODUCTION

The last years a considerable number of numerical water wave models have been developed based on Boussinesq equations. Boussinesq equations are derived from Euler equations after their integration over the depth and the assumption of moderately long waves. They are capable to simulate the propagation of non-linear dispersive waves in shallow water. Thus it is possible to describe the combined effects of numerous wave phenomena such as shoaling, refraction, reflection, and diffraction as well as breaking wave propagation (ABBOTT *et al.*, 1978, 1984, MADSEN and WARREN, 1983, KARAMBAS and KOUTITAS, 1992).

Two important scaling parameters are associated with the analysis of dispersive wave theory. One is the non-linearity parameter ϵ defined as the ratio of wave height to the water depth, ϵ=H/d. The other σ^2 is the square of the ratio of the depth d to a characteristic horizontal length of the surface profile (usually taken equal to the wave length L), $\sigma^2=(d/L)^2$. In shallow water ϵ becomes important taking values of order 1 near the breaking point.

The most usual form of Boussinesq equation (PEREGRINE, 1967, 1972), hereinafter referred to as classical, is based on the assumption that ϵ and σ^2 are small, $O(\epsilon)\ll1$ and $O(\sigma^2)\ll1$, and that ϵ is of the same order as σ^2. For horizontal bottom the equations are written:

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$$\frac{\partial\zeta}{\partial t} + \frac{\partial(Uh)}{\partial x} = 0 \tag{1}$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial\zeta}{\partial x} = \frac{d^2}{3}\frac{\partial^3 U}{\partial x^2\partial t} \tag{2}$$

where U is the mean over the depth velocity and ζ the surface elevation and $h=d+\zeta$.

Good results may be obtained using the system of equations (1) and (2), with the help of an accurate numerical scheme, without the need of inclusion any additional non linear terms (YASUDA *et al.*, 1982, KARAMBAS *et al.*, 1990, KARAMBAS and KOUTITAS, 1992).

The value of the dispersion parameter σ^2 is generally supposed to be small. In deeper water σ^2 becomes important and Boussinesq equations are not valid. For linear waves ($O(\epsilon)=0$) the dispersion relation derived from equations 1 and 2 (PEREGRINE, 1972) is:

$$\omega^2 = \frac{gdk^2}{1 + (kd)^2/3} \tag{3}$$

where $\omega = 2\pi/T$ and $k = 2\pi/L$.

Equation (3) agrees with the exact analytical expression from Airy wave theory (ω^2=gk tanh kd) for small values of kd. Greater values of kd give significant variations of (3) from exact linear relation. Thus Boussinesq equations are restricted to shallow water, with a phase speed error of about 4%, if the limit is extended to d/Lo=0.2 (Lo is the deep water length). On the other hand a system of equations valid at any water depth is the time dependent mild slope equation (ITO and TANIMOTO, 1972, COPELAND, 1985, WATANABE and MARUYAMA, 1986). However, the linear form of the equations is a significant disadvantage.

WITTING (1984) proposed a different form of the Boussinesq equations with improved linear dispersion characteristics, introducing a new velocity variable to replace the bottom horizontal velocity. Similarly, MADSEN *et al.* (1991), NWOGU (1993), STEFFLER and JIN (1993) and SCHAFFER and MADSEN (1995), proposed a different approach with improved linear dispersion characteristics. The new dispersion relation (especially in the latter work) has much smaller discrepancies from the exact linear one, but the two expressions are not identical. In addition the horizontal velocity (u) and pressure (p) distribution over the depth are still parabolic as in the classical Boussinesq equations (PEREGRINE, 1972). In the above models nonlinear terms of order $O(\epsilon\sigma^2)$ have been neglected and the equations are limited to describe weakly nonlinear wave popagation. WEI *et al.* (1995) proposed a fully nonlinear extension of NWOGU (1993) equations with significant improvement in the results concerning solitary wave shoaling and undular bore propagation.

The numerical solution of the above Boussinesq equations (both classical and extended) is based on third-order accuracy Finite Difference schemes (ABBOTT *et al.*, 1984, KARAMBAS *et al.*, 1990, NWOGU, 1993, WEI and KIRBY, 1995).

This work describes an extension of Boussinesq equations in deeper water for monochromatic, non-linear dispersive waves. An extension to irregular waves will be presented in a forthcoming work. The resulting equations are in the same form as equations (1) and (2) but the coefficient ⅓ in the dispersion term is replaced by a coefficient A which is a function of the period T and the total depth h. The vertical distribution of the vertical and horizontal velocities as well as pressure are also derived during the procedure. The new equations are formulated in section 2, the numerical solution is presented in section 4 and the results in section 5.

DERIVATION OF THE EQUATIONS

The Exact Governing Equations

The continuity equation (1), in terms of the elevation ζ and the mean over the depth horizontal velocity U, is an exact relation valid in deep, intermediate and shallow water, for non-linear waves, without any restriction in non-linearity.

Another exact relation, derived from the dynamic free surface boundary condition, is the "conservation of the velocity" law, given by MCDONALD and WITTING (1984):

$$\frac{\partial q_s}{\partial t} = -\frac{\partial}{\partial x}\left[\frac{p_s}{\rho} + g\zeta - \frac{u_s^2 + w_s^2}{2} + u_s q_s\right] \quad (4)$$

where $q_s = u_s + \partial\zeta/\partial\chi\, w_s$, u_s and w_s, the horizontal and vertical velocity at the surface respectively and p_s the pressure at the surface.

The conservation of the velocity law reduces to Bernoulli's equation evaluated at the wave surface for irrotational flow i.e. the dynamic boundary condition (MCDONALD and WITTING, 1984).

The kinematic boundary condition at the surface is expressed by:

$$w = \frac{\partial\zeta}{\partial t} + u_s\frac{\partial\zeta}{\partial x} \quad \text{for } z = \zeta \quad (5)$$

and the boundary condition at the bottom:

$$w = 0 \quad \text{for } z = -d \quad (6)$$

Finally, considering the velocity potential ϕ, the Laplace equation is written:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad (7)$$

where for irrotational flow $u=\partial\phi/\partial x$ and $w=\partial\phi/\partial z$.

A Series Expansion and Approximations

A method of derivation of the Boussinesq equations in shallow (PEREGRINE, 1972) and in deeper water (NWOGU, 1993) is based on the assumption that w varies linearly over the depth. This is not valid in intermediate water where a hyperbolic cosine variation must be used. In this paragraph the vertical variation of w should be expressed in terms of the mean over the depth horizontal velocity U based on an analytical solution of the Laplace equation.

Let's assume the following expansion of the vertical velocity w in a power series of d + z:

$$w = -\left(A_1(d+z) + A_2\frac{(d+z)^2}{h} + A_3\frac{(d+z)^2}{h^3} + \ldots\right)\frac{\partial U}{\partial x} \quad (8)$$

with $A_1 + A_2 + A_3 + \ldots = 1$.

Equation (8) satisfies the boundary conditions:

$$w = 0 \quad \text{for } z = -d$$

and since $A_1 + A_2 + A_3 + \ldots = 1$:

$$w = -h\frac{\partial U}{\partial x} = \frac{\partial\zeta}{\partial t} + U\frac{\partial\zeta}{\partial x} \cong \frac{\partial\zeta}{\partial t} + u_s\frac{\partial\zeta}{\partial x} \quad \text{for } z = \zeta$$

using continuity equation (1) and supposing that $U\partial\zeta/\partial x \cong u_s\partial\zeta/\partial x$. That should be, as far as the non-linear order of equation is concerned, a restriction of the present derivation (see the scaling analysis in the next paragraph).

EAGLESON and DEAN (1966) used the method of separation of variables for the solution of the Laplace equation. They assumed that the velocity potential ϕ is written in a the product form:

$$\phi(x,z,t) = X(x)\, Z(z)\, T(t)$$

where X(x), Z(z) and T(t) are functions of x, z and t respectively, and derived the following solution:

$$\phi(x,z,t) = X(x)\,(C\,e^{kz} + D\,e^{\,kz})\,T(t) \quad (9)$$

where C and D are constants and k the wave number.

Their periodic solution (of the linear Laplace equation) for linear progressive waves based on four separate elementary combinations of the above solution (9) with different X(x) and T(t) functions and the same Z(z) function. However equation (9) can be reached without using the small amplitude assumption. The same conclusion i.e. the distribution Z(z) is also valid for non linear waves, is pointed out also by KINSMAN (1984, p. 249) who derived a Stokes 3rd order theory based on equation (9). The next approach is based only on the z-dependent part Z(z) of the solution (9).

The application of the boundary condition (7) gives:

$$\phi = Ee^{kd}\cosh k(d+z) \quad (10)$$

with $E = 2 \times X(x)DT(t)$ for the velocities, u and w:

$$w = \frac{\partial \phi}{\partial z} = KEe^{kd}\sinh k(d+z)$$

$$u + \frac{\partial \phi}{\partial x} = \frac{\partial E}{\partial x}e^{kd}\cosh k(d+z) \quad (11)$$

After the integration over the depth of u in (11), to derive U, it can be easily found that:

$$w = -h\frac{\sinh k(d+z)}{\sinh kh}\frac{\partial U}{\partial x} \quad (12)$$

which is similar to equation (8).

In terms of power series of d + z (12) becomes:

$$w = -\left(\frac{kh}{\sinh kh}\frac{(d+z)}{1} + \frac{(kh)^3}{3!\sinh kh}\frac{(d+z)^3}{h^2} + \ldots\right)\frac{\partial U}{\partial x} \quad (13)$$

The comparison of (13) with (8) gives the expression of the coefficient A_n:

$$A_n = \frac{(kh)^n}{n!\sinh kh} \quad \text{for } n = 1, 3, 5, 7, \ldots$$

$$A_n = 0 \quad \text{for } n = 2, 4, 6, 8, \ldots \quad (14)$$

with $A_1 + A_3 + A_5 \ldots = 1$, as had been supposed.

Scaling Analysis

The traditional way in the derivation of Boussinesq-type equations is to integrate the local momentum equations over the depth (PEREGRINE, 1967, NWOGU, 1993). A more direct derivation has been proposed by MEI (1983) who derived the equations using the free surface surface boundary condition. In the next a similar procedure is adopted.

In order to make more clear to what order in ϵ and σ the approximations are being made a non-dimensional form of the equations is derived. The dependent and independent variables are scaled as follows (NWOGU, 1993):

$$x' = xL \qquad z' = zd \qquad t' = tL/c \qquad u' = u\frac{Hc}{d}$$

$$\zeta' = \zeta H \qquad w' = w\frac{cHL}{d^2} \quad \text{with } c = (gd)^{1/2}$$

here "′" indicate temporarily a dimensional variable.

Equation (13) is based on an analytical solution of the Laplace equation which is adequate for a third order Stokes non-linear wave theory (KINSMAN, 1984, p. 248–251) (in the case of linear waves is valid for arbitrary σ^2 in intermediate and shallow water). In (13) the following approximation is also adopted:

$$U\partial\zeta/\partial_x \cong u_s\partial\zeta/a_x$$

Using the above definitions the kinematic boundary condition at the surface can be expressed in nondimensional form (see also NWOGU, 1993):

$$w_s = \sigma^2\frac{\partial \zeta}{\partial t} + \epsilon\sigma^2 u_s\frac{\partial \zeta}{\partial x} \quad \text{for } z = \epsilon\zeta$$

or, using continuity equation (1):

$$w_s = -\sigma^2(1+\epsilon\zeta)\frac{\partial U}{\partial x} + \epsilon\sigma^2(u_s - U)\frac{\partial \zeta}{\partial x} \quad (15)$$

From to the works of PEREGRINE (1972) and MEI (1983, p. 508) it can be easily derived that:

$$u_s - U = O(\sigma^2)$$

Thus, by replacing U with u_s in the above, the error is of order $O(\epsilon\sigma^4)$. This error is introduced through equation (8) in the form of Boussinesq equation proposed here.

The flow is supposed to be irrotational, so:

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}$$

Replacing w from equation (8) or (13) into the above we have, after integrating with respect to z:

$$u = U - \sigma^2\left[\left(A_1\frac{(d+z)^2}{2} + A_3\frac{(d+z)^4}{4(d+\epsilon\zeta)^2} + \ldots\right) - (d+\epsilon\zeta)^2\left(\frac{A_1}{6} + \frac{A_3}{20} + \ldots\right)\right]\frac{\partial^2 U}{\partial x^2} \quad (16)$$

The horizontal velocity u_s at the surface is given by:

$$u_s = U - \sigma^2 A(d+\epsilon\zeta)^2\frac{\partial^2 U}{\partial x^2} \quad (17)$$

with $A = A_1/3 + A_3/5 + A_5/7 \ldots$

Equation (4) is expressed in nondimensional form as:

$$\sigma^2\frac{\partial u_s}{\partial t} + \epsilon\sigma^2 w_s\frac{\partial^2 \zeta}{\partial x\partial t} + \epsilon\sigma^2\frac{\partial \zeta}{\partial x}\frac{\partial w_s}{\partial t} = -\sigma^2\frac{\partial \zeta}{\partial x} - \epsilon\sigma^2 u_s\frac{\partial u_s}{\partial x} + \epsilon w_s\frac{\partial w_s}{\partial x} \quad (18)$$

containing only terms of order $O(\epsilon)$, $O(\sigma^2)$ and $O(\epsilon\sigma^2)$.

The substitution of equations (15) and (17) into equation (18) retaining terms up to $O(\epsilon)$, $O(\sigma^2)$ and $O(\epsilon\sigma^2)$, leads to the derivation of the momentum equation:

$$\frac{\partial U}{\partial t} + \epsilon U\frac{\partial U}{\partial x} + \frac{\partial \zeta}{\partial x} = \sigma^2(d+\epsilon\zeta)^2 A\frac{\partial^3 U}{\partial x^2\partial t} + \epsilon\sigma^2 A(d+\epsilon\zeta)^2\left(U\frac{\partial^3 U}{\partial x^3} - \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x^2}\right) + \epsilon\sigma^2(d+\epsilon\zeta)\frac{\partial \zeta}{\partial x}\frac{\partial^2 U}{\partial x\partial t} \quad (19)$$

In a dimensional form is written:

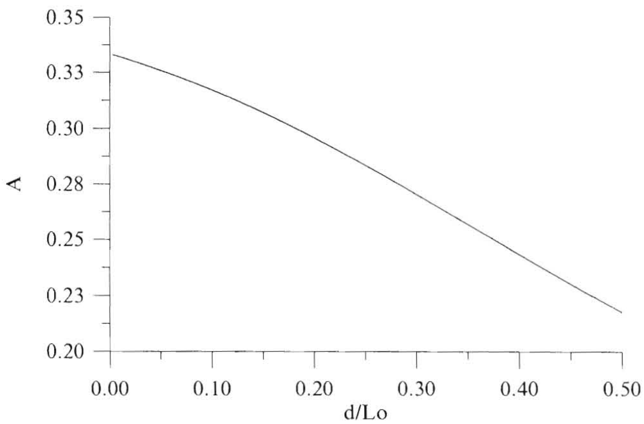

Figure 1. Variation of coefficient A with d/Lo (linear waves).

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial \zeta}{\partial x}$$

$$= Ah^2\frac{\partial^3 U}{\partial x^2 \partial t} + Ah^2\left(U\frac{\partial^3 U}{\partial x^3} - \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x^2}\right) + h\frac{\partial \zeta}{\partial x}\frac{\partial^2 U}{\partial x \partial t} \quad (20)$$

with $h=d+\zeta$.

The above equation is similar to the Serre equation (SERRE, 1953, MEI, 1983, DINGEMANS, 1997).

The coefficient A is given by:

$$A = A_1/3 + A_3/5 + A_5/7 \ldots$$

where A_n is calculated from (14) and k from the linear wave theory. Thus A is a function of the depth h and the period T (Figure 1, for linear waves). For depths smaller than half of the deep water length, a correct estimation of A requires less than seven terms of the sum. In shallow water $kh \to 0$, $A \to 1/3$ and equation (20) becomes identical to the classical Serre equation. Retaining terms up to $O(\epsilon)$ and $O(\sigma^2)$ equation (20) becomes identical to the classical Boussinesq equation (2).

The velocity distribution (eq. 16) is written in dimensional variables:

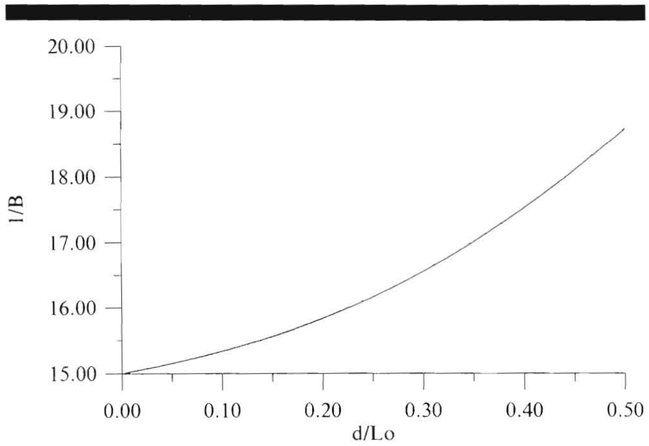

Figure 2. Variation of coefficient B with d/Lo (linear waves).

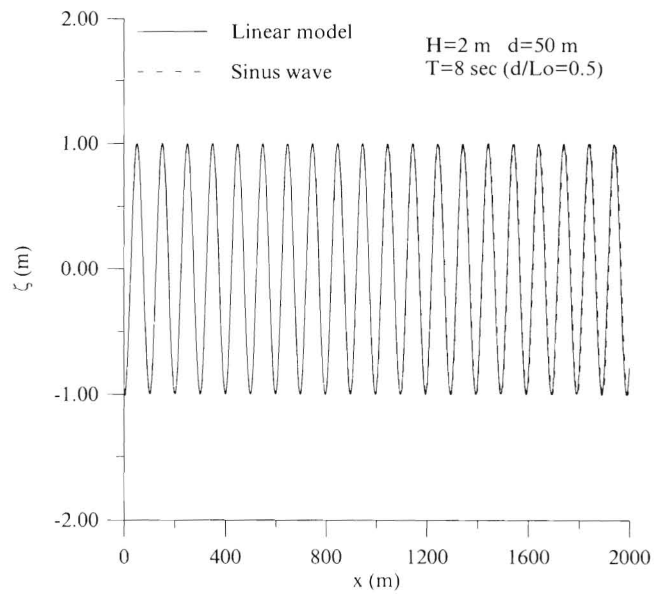

Figure 3. Sinusoidal wave propagation, using a linear version of the model i.e. excuding non-linear terms.

$$u = U - \left[\left(A_1\frac{(d+z)^2}{2} + A_3\frac{(d+z)^4}{4(d+\zeta)^2} + \ldots\right) - (d+\zeta)^2\left(\frac{A_1}{6} + \frac{A_3}{20} + \ldots\right)\right]\frac{\partial^2 U}{\partial x^2}$$

The pressure distribution, although is not used in the present derivation, is obtained considering the z-momentum equation (see NOWOGU, 1993):

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} + g = 0$$

By substituting equation (13) into the above and integrating with respect to z:

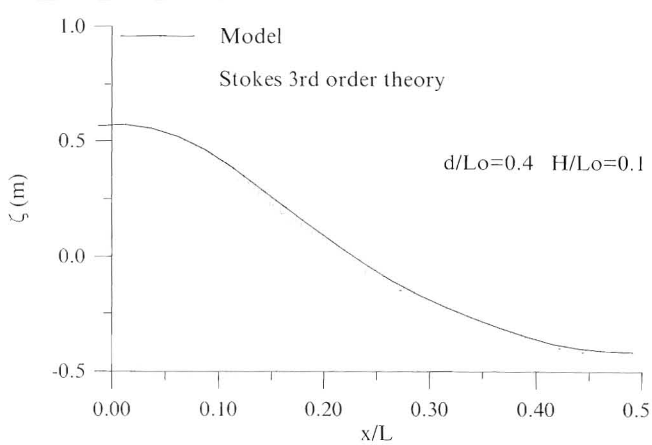

Figure 4. Surface elevation of a non-linear wave in intermediate water: Comparison with Stokes III theory (d/Lo = 0.4, H/d = 0.25).

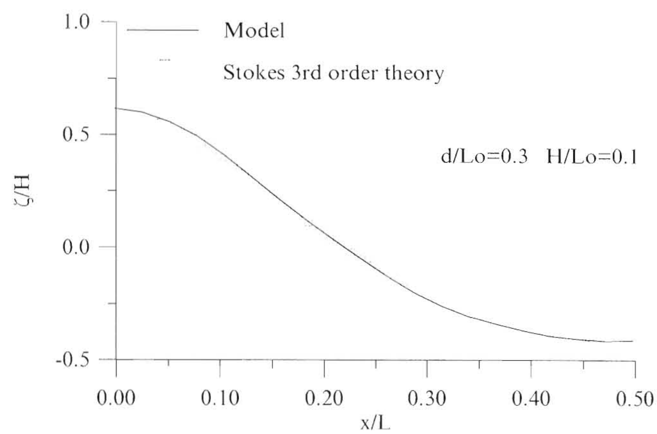

Figure 5. Surface elevation of a non-linear wave in intermediate water: Comparison with the Stokes III theory (d/Lo = 0.3, H/d = 0.333).

$$p = \rho g(\zeta - z) + \rho F(d + z)\frac{\partial^2 U}{\partial x \partial t}$$

where

$$F(d + z) = \left(A_1\frac{(d + z)^2}{2} + A_3\frac{(d + z)^4}{4h^2} + \ldots\right) - h^2\left(A_1\frac{1}{2} + A_3\frac{1}{4} + \ldots\right)$$

In shallow water, where $A_1 \rightarrow 1$ and $A_{3,5,\ldots} \rightarrow 0$, both distributions become identical to the parabolic ones, obtained by PEREGRINE (1972) for the classical Boussinesq equations.

Uneven Bottom

For the case of an uneven bottom we also need an expression for the vertical velocity w(z). Adopting the same power expansion as the case for an horizontal bottom (*i.e.* equation (8)) we have:

$$w(z) = w_o + (w_s - w_o)\frac{G(d + z)}{h},$$

$$G(d + z) = A_1(d + z) + A_3\frac{(d + z)^3}{h^2} + \ldots \qquad (21)$$

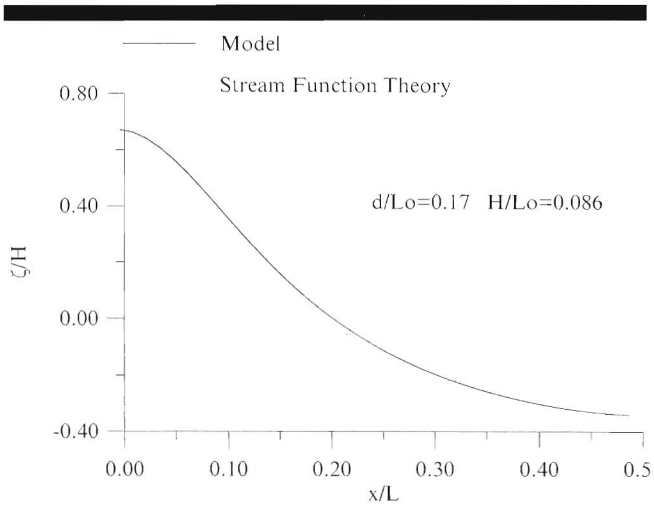

Figure 6. Surface elevation of a non-linear wave in shallow water: Comparison with the Stream Function theory (d/Lo = 0.17, H/d = 0.5).

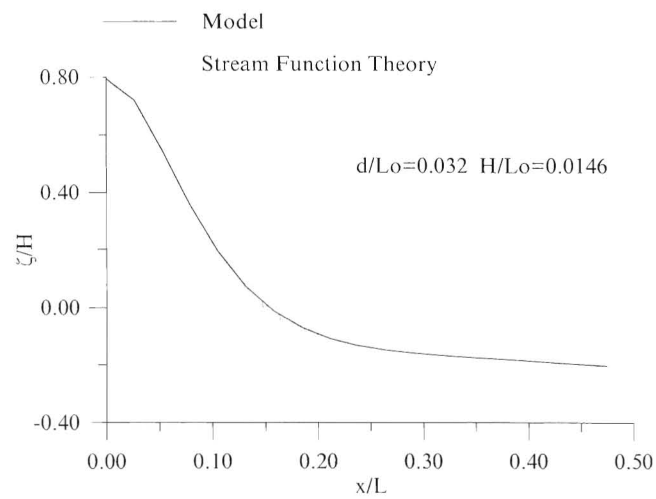

Figure 7. Surface elevation of a non-linear wave in shallow water: Comparison with the Stream Function theory (d/Lo = 0.032, H/d = 0.45).

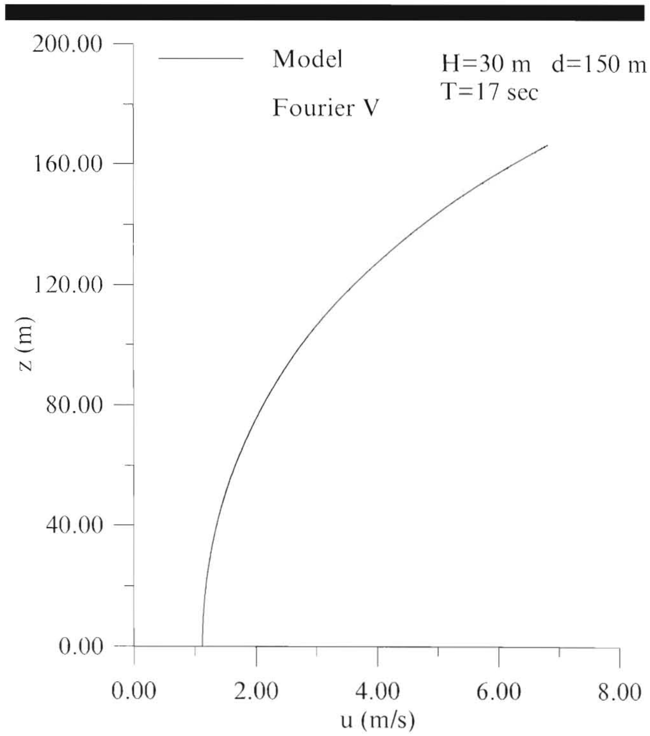

Figure 8. Vertical distribution of the horizontal velocity u(z) in intermediate water: Comparison with the Fourier theory.

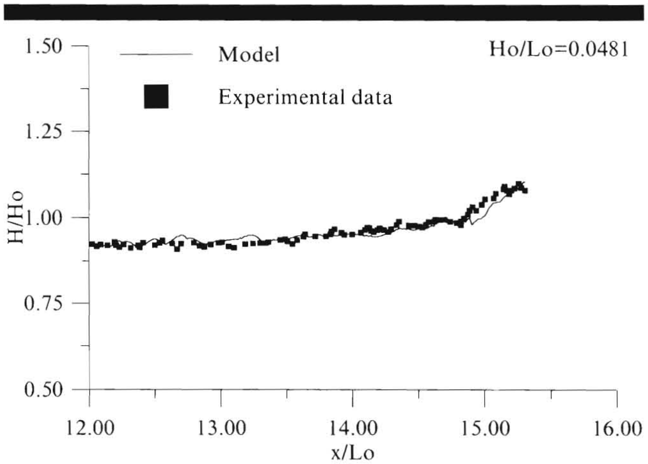

Figure 9. Wave shoaling on a 1:34.26 slope. Comparison between model results and experimental data by Buhr Hansen and Svendsen (1979). Test no A10112 (H = 70 mm and frequency F = 1 Hz).

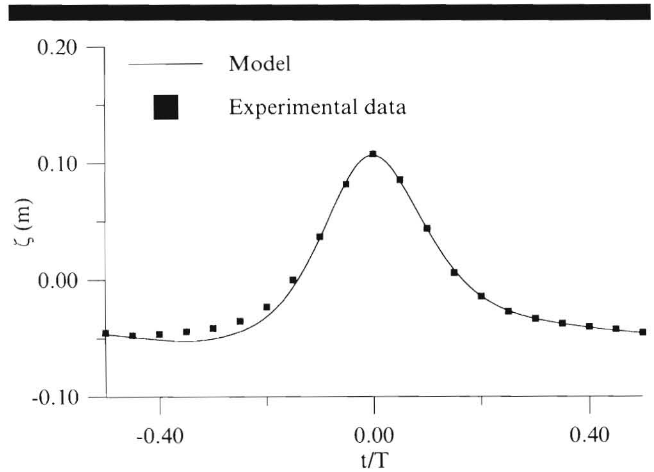

Figure 11. Comparison between model results and experimental data (Stive, 1983) of free surface elevation at the point x = 32.5 m. Test 1 (H = 14.51 cm and T = 1.79 sec).

where the coefficients A_n are given from the equation (14) and w_o and w_s are the vertical velocities at the bottom and surface respectively:

$$w_o = -\sigma^2 d_x u_o \quad \text{for } z = \zeta$$

in which d_x is the bottom slope (a slowly-varying bathymetry is assumed) and u_o is the horizontal velocity at the bottom (given by equation (16) for z = − d), and

$$w_s = -\sigma^2\left(h\frac{\partial U}{\partial x} + d_x U\right) \quad \text{for } z = -d \tag{22}$$

with $h = d + \epsilon\zeta$.

Using again the continuity equation (1) and supposing, as in the previous paragraph, that $U\partial\zeta/\partial x \cong u_s\partial\zeta/\partial x$, equation (22) becomes:

$$w_s = \sigma^2\left(\frac{\partial\zeta}{\partial t} + U\frac{\partial\zeta}{\partial x}\right) \cong \sigma^2\left(\frac{\partial\zeta}{\partial t} + u_s\frac{\partial\zeta}{\partial x}\right)$$

In this way equation (21) satisfies both the bottom and the surface boundary conditions.

In the next we follow the same procedure as in the case for horizontal bottom. The flow is supposed to be irrotational, so:

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}$$

Replacing w from equation (21) into the above we have, after integration with respect to z:

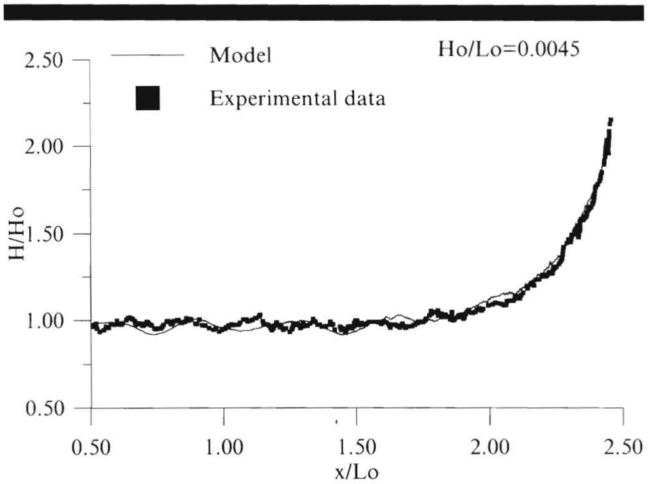

Figure 10. Wave shoaling on a 1:34.26 slope. Comparison between model results and experimental data by Buhr Hansen and Svendsen (1979). Test no 041041 (H = 40 mm and F = 0.4 Hz).

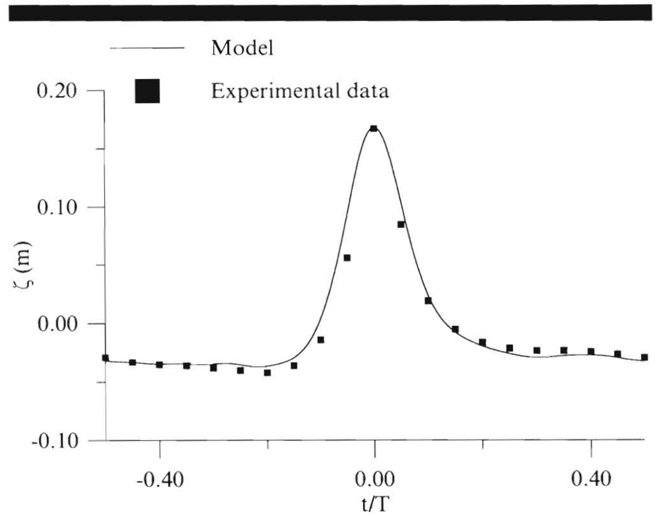

Figure 12. Comparison between model results and experimental data (Stive, 1983) of free surface elevation at the point x = 32.5 m. Test 2 (H = 14.43 cm and T = 3.0 sec).

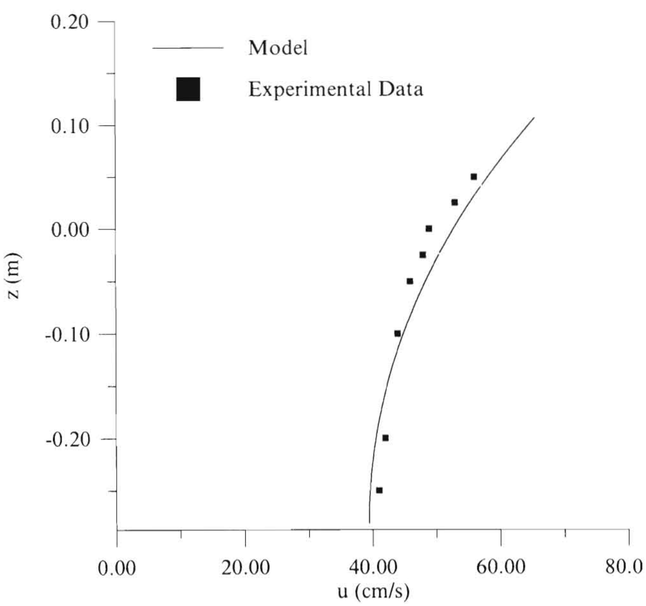

Figure 13. Comparison between model results and experimental data (Stive, 1983) of horizontal velocity distribution under the crest at the point x = 32.5 m. Test 1 (H = 14.51 cm and T = 1.79 sec).

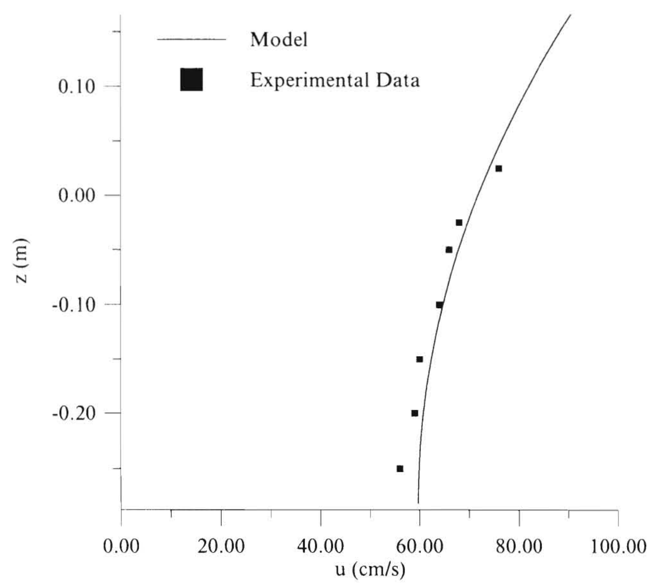

Figure 14. Comparison between model results and experimental data (Stive, 1983) of horizontal velocity distribution under the crest at the point x = 32.5 m. Test 2 (H = 14.43 cm and T = 3.0 sec).

$$\begin{aligned} u(z) = U &- \sigma^2 \frac{\partial D}{\partial x}\left[\left(A_1 \frac{(d+z)^2}{2} + A_3 \frac{(d+z)^4}{4h^2} + \dots\right) - \left(A_1 \frac{h^2}{6} + A_3 \frac{h^4}{20} + \dots\right)\right] \\ &- \sigma^2 D\left[\left(A_{1,x} \frac{(d+z)^2}{2} + A_{3,x} \frac{(d+z)^4}{4h^2} + \dots\right) - \left(A_{1,x} \frac{h^2}{6} + A_{3,x} \frac{h^4}{20} + \dots\right)\right] \\ &- \sigma^2 d_x D\left[\left(A_1 z + A_3 \frac{(d+z)^3}{h^2} + \dots\right) - A_1 \zeta + h\left(A_1 \frac{1}{2} - A_3 \frac{1}{4} - \dots\right)\right] \\ &- \sigma^2 d_x \frac{\partial u_o}{\partial x} z - \sigma^2 d_x \frac{\partial u_o}{\partial x} \frac{h}{2} \end{aligned} \tag{23}$$

where the subscript x denotes a partial derivative and D is given by:

$$D = \frac{\partial U}{\partial x} + d_x \frac{U - u_o}{h}$$

where $U - u_o = O(\sigma^2)$

The horizontal velocity at the surface is written:

$$u_s = U - \sigma^2 h^2 \frac{\partial(AD)}{\partial x} - \sigma^2 d_x h\left[D\left(A_1 \frac{1}{2} + A_3 \frac{3}{4} + \dots\right) + \frac{\partial u_o}{\partial x} \frac{1}{2}\right] \tag{24}$$

The substitution of equations (22) and (23) into equation (18), retaining terms up to $O(\epsilon)$, $O(\sigma^2)$ and $O(\epsilon\sigma^2)$, leads to the derivation of the momentum equation over uneven bottom (in variables with dimension):

$$\begin{aligned} &\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial \zeta}{\partial x} \\ &= h^2 \frac{\partial^2\left(A\frac{\partial U}{\partial x}\right)}{\partial x \partial t} + Ah^2\left(U\frac{\partial^3 U}{\partial x^3} - \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x^2}\right) + d\frac{\partial \zeta}{\partial x}\frac{\partial^2 U}{\partial x \partial t} \\ &\quad + d_x h\left[\frac{\partial^2 U}{\partial x \partial t}\left(A_1 \frac{1}{2} + A_3 \frac{3}{4} + \dots\right) + \frac{\partial^2 u_o}{\partial x \partial t}\frac{1}{2}\right] \\ &\quad + d_x hU \frac{\partial^2 U}{\partial x^2}\left[\left(A_1 \frac{1}{2} + A_3 \frac{3}{4} + \dots\right) + \frac{1}{2}\right] + d_x \frac{\partial \zeta}{\partial x}\frac{\partial U}{\partial t} \end{aligned} \tag{25}$$

As mentioned before, in the previous derivations a slowly-varying bathymetry is assumed and consequently terms with d_{xx} and $(d_x)^2$ have been ignored.

A Different Form of the Equations

MADSEN *et al.* (1991) derived a new form of the Boussinesq equations by adding a small quantity of order $O(\epsilon\sigma^2, \sigma^4)$ to the momentum equation (2) (see also SCHAFFER and MADSEN, 1995). Based on the same procedure, but using the non-linear long wave equation instead of the linear one, we consider:

$$\frac{\partial U}{\partial t} + \epsilon U\frac{\partial U}{\partial x} + \frac{\partial \zeta}{\partial x} = O(\sigma^2)$$

which leads to the following small quantity:

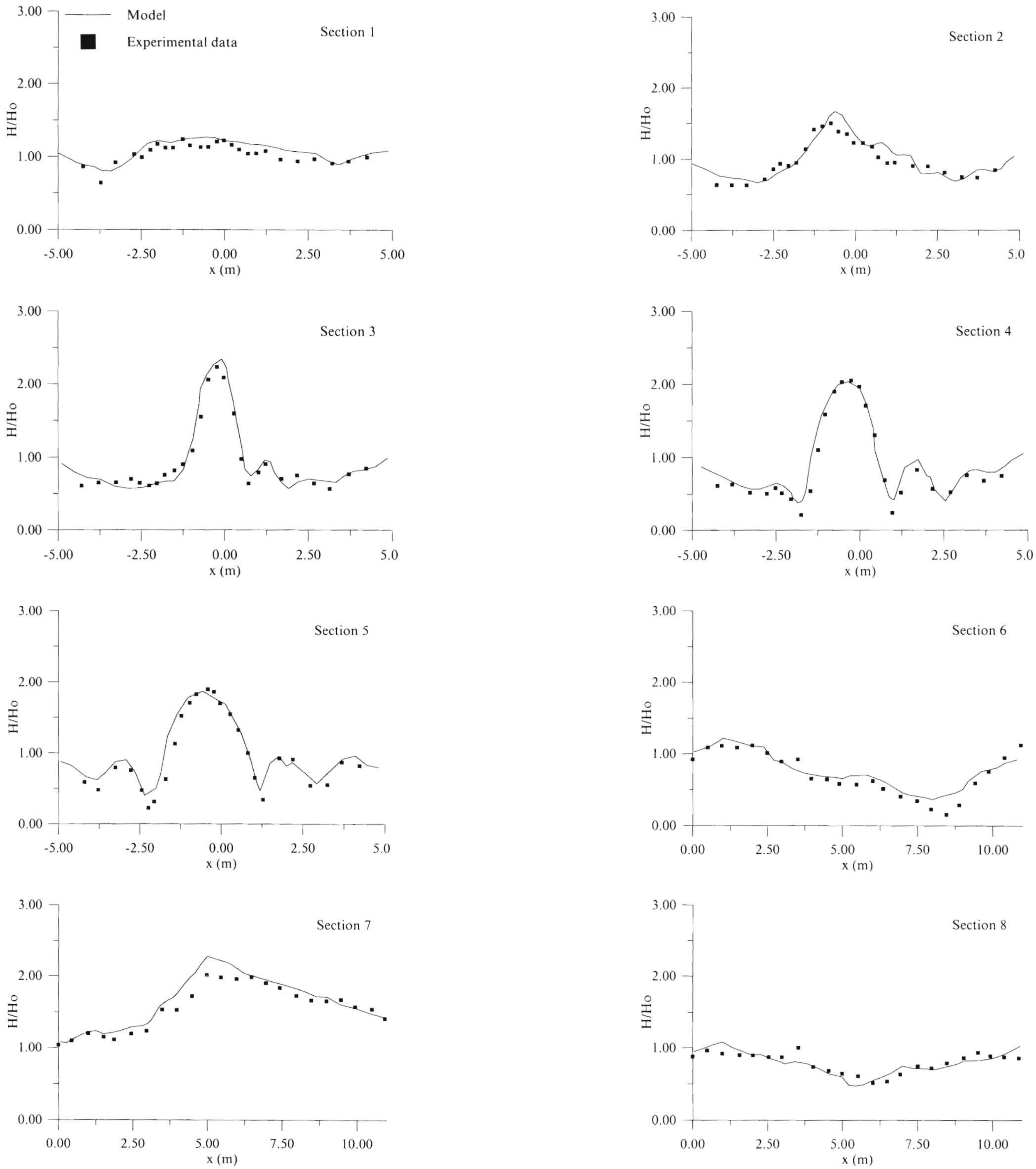

Figure 15. Comparison of numerical results and experimental data for wave heights on measured transects of Berkhoff *et al.* (1982).

$$-B\sigma^2 d^2\left(\frac{\partial^3 U}{\partial x^2 \partial t} + \frac{\partial^3 \zeta}{\partial x^3} + \epsilon\frac{\partial^2\left(U\frac{\partial U}{\partial x}\right)}{\partial x^2}\right) = O(\sigma^4) \qquad (26)$$

with B a coefficient. The above small quantity differs from that used by MADSEN *et al.* (1991) by the non linear term $\epsilon\sigma^2(UU_x)_{xx}$ (because in the present work terms of order $O(\epsilon\sigma^2)$ are not neglected). The quantity is added to (25) (in which terms of order $O(\sigma^4)$ have been neglected) resulting to a new form of Boussinesq equations. The same procedure was also adopted by SCHAFFER and MADSEN (1995) in order to derived another form of the equations.

The momentum equation is now written in the form:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + g\frac{\partial \zeta}{\partial x}$$
$$= h^2\frac{\partial^2\left(A\frac{\partial U}{\partial x}\right)}{\partial x \partial t} + Ah^2\left(U\frac{\partial^3 U}{\partial x^3} - \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x^2}\right) + d\frac{\partial \zeta}{\partial x}\frac{\partial^2 U}{\partial x \partial t}$$
$$+ d_x h\left[\frac{\partial^2 U}{\partial x \partial t}\left(A_1\frac{1}{2} + A_3\frac{3}{4} + \ldots\right) + \frac{\partial^2 u_0}{\partial x \partial t}\frac{1}{2}\right]$$
$$+ d_x hU\frac{\partial^2 U}{\partial x^2}\left[\left(A_1\frac{1}{2} + A_3\frac{3}{4} + \ldots\right) + \frac{1}{2}\right] + d_x\frac{\partial \zeta}{\partial x}\frac{\partial U}{\partial t}$$
$$+ Bd^2\left(\frac{\partial^3 U}{\partial x^2 \partial t} + g\frac{\partial^3 \zeta}{\partial x^3} + \frac{\partial^2\left(U\frac{\partial U}{\partial x}\right)}{\partial x^2}\right) \qquad (27)$$

Following the same procedure the two-deminsional form of the equations is obtained:

$$\mathbf{U}_t + \mathbf{U}\nabla\mathbf{U} + g\nabla\zeta$$
$$= h^2\nabla(A\nabla\mathbf{U})_t + Ah^2(\mathbf{U}\nabla^3\mathbf{U} - \nabla\mathbf{U}\nabla^2\mathbf{U}) + d\nabla\zeta\nabla\mathbf{U}_t$$
$$+ \nabla dh\left[\nabla\mathbf{U}_t\left(A_1\frac{1}{2} + A_3\frac{3}{4} + \ldots\right) + \nabla\mathbf{u}_{0,t}\frac{1}{2}\right]$$
$$+ \nabla dh\left[\mathbf{U}\nabla^2\mathbf{U}\left(A_1\frac{1}{2} + A_3\frac{3}{4} + \ldots\right) + \frac{1}{2}\right] + \nabla d\nabla\zeta\nabla\mathbf{U}$$
$$+ Bd^2(\nabla^2\mathbf{U}_t + g\nabla^3\zeta + \nabla^2\mathbf{U}\nabla\mathbf{U}) \qquad (28)$$

In order to obtain the same linear dispersion relation with that corresponding to equation (20) the following procedure is adopted for the estimation of the coefficient B. The comparison of the momentum equation (20) (including linear terms only) and the above equation (27) leads to:

$$(B + 1/3)\, d^2\, U_{xxt} + B\, g\, d^2\, \zeta_{xxx} = A\, d^2 U_{xxt}$$

where $F_x \equiv \partial F/\partial x$.

Replacing ζ_{xxx} and U_{xxt} from linear wave theory (since only linear terms of order $O(\sigma^2)$ are involved) the value of B is given by the relation:

$$B = -\frac{A - 1/3}{A(kd)^2} \qquad (29)$$

In the next the above form of the momentum equation should be used in the numerical computations. The reason which led us to a such a transformation of the original equations is outlined below. Applying equation (27) in deep water, the wave energy is propagated into the model without the significant dispersive behaviour of the front which is present in the classical form (equation 25)—see also MADSEN *et al.*, (1991). In addition, in periodic non linear waves propagating in shallow water over a constant depth, higher harmonics are bounded and travel with the velocity of the basic wave and since coefficient A is a function of the first harmonic only, equation (25) can be used. In the contrary when a first order boundary condition in shallow water is applied or the waves travel over slopes and bars, more higher harmonics are generated propagating as free components (CHAPALAIN *et al.*, 1992, DINGEMANS, 1993). In this case equation (27) is used, which can provide significant improvement of the dispersion relation of the higher free harmonics (MADSEN and SORENSEN, 1993) with respect to the equation (25), while using equation (29) for B (instead of the constant value 1/15) the propagation of the first harmonic is still predicted accurately.

Figure 2 shows the variation of 1/B with the ratio d/Lo, for linear waves. For small values of d/Lo (and kd) $1/B \rightarrow 15$ as proposed in MADSEN and SORENSEN (1992).

In this way the new type of equations becomes similar to the MADSEN *et al.* (1991) and NWOGU (1993) models, with the new value of B, valid for monochromatic only waves. Since the two dispersion relations corresponding to the new versions of the Boussinesq equations (eq. 25 and eq. 27) are also identical and equal to the exact linear one (see the next paragraph), the new model has now a more accurate form as far as the linear dispersion characteristics are concerned.

The models proposed by MADSEN *et al.* (1991) and NWOGU (1993), have the advantage of being able to simulate irregular wave propagation in intermediate water. The present model can also be used for the simulation of non monochromatic waves adopting a mean period for the calculation of B. In this case the value of B should be between 1/15 and 1/18.6 (Figure 2).

RANGE OF APPLICATION OF THE NEW EQUATIONS

The only difference between the new system of equations and the classical type of Serre equations is the coefficient A (and B) in the linear dispersion term. It is reminded here that the whole procedure is based on a solution of the linear Laplace equation which gave the distribution of the vertical velocity w. As a result the linear dispersion relation corresponding to equation (25):

$$\omega^2 = \frac{gdk^2}{1 + A(kd)^2} \qquad (30)$$

Substitution of the values of A in (30) yields:

$$\omega^2 = \frac{gkkd \sinh kd}{\sinh kd + \left(\frac{kd}{3\,1!} + \frac{(kd)^3}{5\,3!} + \frac{(kd)^5}{7\,5!} + \ldots\right)(kd)^2}$$

or

$$\omega^2 = \frac{gk \sinh kd}{1 + \frac{(kd)^2}{2!} + \frac{(kd)^4}{4!} + \ldots} = gk\frac{\sinh kd}{\cosh kd} = gk \tan kd \qquad (31)$$

the well known relation from Airy linear theory.

Thus, as far as the linear dispersion properties are concerned, the equations are exact.

In the proposed set of non-linear equations the continuity equation, which is exact in deep, intermediate and shallow water without any restriction in non-linearity (*i.e.* to all orders of ϵ and σ^2), remains unchanged. The equations are derived keeping terms of order up to $O(\epsilon\sigma^2)$ in the momentum equation. Considering the above derivation, as a deep water limit of the validity of equations can be considered the practical deep water limit i.e. the half of the wave length, $d/L = 0.5$. For a near breaking wave in intermediate water, say $\sigma = d/L = 0.5$, the value of ϵ is about $\epsilon = 0.28$ (considering a breaking limit $H/L = 0.14$) and the product $\epsilon\sigma^2 = 0.07$ (a small quantity). Thus the derived set of equations is able to simulate non-linear dispersive wave propagation in intermediate and shallow water.

NUMERICAL SOLUTION

The numerical method is based on a third order accuracy scheme, the principles of which have been described by KARAMBAS *et al.* (1990) and KARAMBAS (1991). In the next only the differences between to present numerical code and the existing one will be emphasized.

The partial derivatives of the set of equations (1) and (24) can be approximated using central finite differences both in space and time:

$$\left.\frac{\partial F}{\partial t}\right|_i^n \simeq \frac{F_i^{n+1} - F_i^{n-1}}{2\Delta t} \qquad \left.\frac{\partial F}{\partial x}\right|_i^n \simeq \frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x}$$

where F is the velocity U or the elevation ζ.

In order to overcome the numerical dispersion and dissipation some correction terms have to be included in the F.D. integration. These terms are detected by the Taylor series expansion of the significant terms:

$$\left.\frac{\partial F}{\partial t}\right|_i^n \simeq \frac{F_i^{n+1} - F_i^{n-1}}{2\Delta t} - \frac{\Delta t^2}{6}\frac{\partial^3 F}{\partial t^3}$$
$$\left.\frac{\partial F}{\partial x}\right|_i^n \simeq \frac{F_{i+1}^n - F_{i-1}^n}{2\Delta x} - \frac{\Delta x^2}{6}\frac{\partial^3 F}{\partial x^3} \qquad (32)$$

In the existing numerical schemes the above correction terms have been transformed with the help of the linear long wave equations to be similar to the dispersion term (see also ABBOT *et al.*, 1984). Obviously since linear long wave equations are not valid in deep water, these should not be used here. Instead of this the simplified form of the time-dependent mild-slope equation (COPELAND, 1985) is used together with the linear continuity equation:

$$\frac{\partial U}{\partial t} + \frac{c^2}{d}\frac{\partial \zeta}{\partial x} = 0$$
$$\frac{\partial \zeta}{\partial t} + d\frac{\partial U}{\partial x} = 0 \qquad (33)$$

Based on the above equations and following the same method for the transformation of the truncation errors the new correction terms become:

$$\left(-\frac{\Delta x^2}{6} + c^2\frac{\Delta t^2}{6}\right)\frac{\partial^3 \zeta}{\partial x^2 \partial t}$$
$$\left(-\frac{gd}{c^2}\frac{\Delta x^2}{6} + c^2\frac{\Delta t^2}{6}\right)\frac{\partial^3 U}{\partial x^2 \partial t} \qquad (34)$$

in the right hand side of the continuity and momentum equation respectively.

As in ABBOT *et al.* (1984) bottom slope effects are not significant and are ignored in the correction terms. The extension in a two-dimensional case (see also NEVES and SILVA, 1988) can be easily introduced adopting a full form of the equations (33).

In shallow water, where $c = (gd)^{1/2}$, equation (33) reduces to the linear long wave equation, and the numerical scheme becomes identical to the one for classical Boussinesq equations. In this way the numerical scheme also maintains the unified form of the system of equations, i.e. their validity both in intermediate and shallow water. In the last region, for values of Δx near d, the correction terms are very important since they are generally of the same order as the non hydrostatic dispersion term of the Boussinesq momentum equation. In deeper water the value of Δx is much smaller than the value of d and the correction terms become less important.

For the 2-D cases exactly the same numerical scheme is used (Karambas, 1991) instead of an ADI algorithm in order to have a symmetric treatment of all variables with respect to x and y. The resulting linear system of equations is solved using a predictor-corrector technique (MATSOUKIS, 1986).

The model is driven at the open boundary by a time function of the free surface $\zeta_i(t)$. The open boundary condition allows for the reflect wave to be radiated out of the computational domain (HAUGEL, 1980, KARAMBAS, 1992):

$$-(d + \zeta)\,\mathbf{U}\,\mathbf{n} + c\,(d + \zeta) = (1 - \mathbf{u}_v\,\mathbf{n})\,c\,\zeta_i(t) \qquad (35)$$

where $\mathbf{n}$ is the unit vector normal to the boundary (positive outwards) and $\mathbf{u}_v$ is the unit vector of the propagation direction.

At the outgoing wave boundary (perfectly absorbing or partial reflection) the sponge layer technique is applied.

The above numerical solution is evaluated, as far as the numerical dispersion errors are concerned, for its ability to propagate a linear wave with correct celerity. The results from the propagation of a sine wave over a constant depth in deep water is presented in Figure 3 using a linear version of the model i.e. excluding the non linear terms. The wave is propagated a long distance without significant changes in its height and its celerity, illustrating in this way the small numerical dispersion error which is introduced in the numerical solution.

APPLICATION

The aim of this study was not to develope a model to replace the non-linear wave theory but to improve the existing numerical models extending their validity in deeper water. However in order to test the computational model a comparison with non-linear theories and experimental data is presented. Stokes 3rd order theory for intermediate water and Stream Function theory for intermediate and shallow water are chosen for the comparison.

Figures 4 and 5 show a comparison of the model results with Stokes III theory concerning the prediction of the surface elevation ζ in intermediate water: d/Lo=0.4 and d/Lo=0.3. The input at the western boundary is a time serie of the surface elevation ζ_i using Stokes III theory. Stokes III waves may not be consistent with the derived equations. However it is a solution valid to indermediate water, close to the nonlinear order which is considered here. In addition with the use of the theory, suppression of the free higher harmonics is obtained. In this way, the well-known spatial variations in wave properties are avoided. After several wave lengths of propagation in a channel with horizontal bottom the instantaneous wave profile, from x = 15L to x = 15.5L, is compared with Stokes III theory. The non-linearity is chosen to be important: H/d = 0.25, H/d = 0.333, and in both cases H/L≃0.10. The numerical results agree reasonably well with the theory. In shallow water the model is tested against Stream Function Theory (CHAPLIN, 1980). At the western boundary a cnoidal wave is used as input and the comparison is made at the position x = 15L to x = 15.5L. The comparison between model results and theory is presented in Figures 6 (d/Lo = 0.17, H/d = 0.5) and Figure 7 (d/Lo = 0.032, H/d = 0.45).

Another significant numerical experiment (Figure 8) is the comparison of the vertical distribution of the horizontal velocity u(z), as predicted by equation (16), with the Fourier non-linear theory (HUANG, 1990). The model predicts well the u distribution over the depth confirming also the validity of the coefficients A_n, upon which the present model has been developed.

In the next the propagation of non-linear dispersive waves over uneven bottom will be considered. Model results are compared with three different experimental set of data. In the first experiment, by BUHR HANSEN and SVENDSEN (1979), the transformation of non-linear waves over a sloping plane bottom is investigated. The experiments were made in a wave flume 60 cm wide and 32 m long. In the upstream part of the flume with horizontal bottom the depth was 36 cm. The toe of the beach was 14.78 m from the wave generator and the slope was 1:34.26. Test no A10112 (H = 70 mm and frequency F = 1 Hz) and 041041 (H = 40 mm and F = 0.4 Hz) are reproduced. In Figures 9 and 10 the model results (wave height) are compared with experimental data. The results agree very well with measurements.

Experimental data for the second comparison are obtained by STIVE (1983). The experiments were conducted in a wave flume 1 m wide and 55 m long. The basic set-up consisted of a plane beach of 1:40 slope. The toe of the beach was 16 m from the wave generator (x = 16 m) and the water depth in the horizontal section was 0.70 m. Two tests are reproduced: test 1 (H = 14.51 cm and T = 1.79 sec) and test 2 (H = 14.43 cm and T = 3.0 sec). Surface elevation and horizontal velocity (under the crest) measurements at the point x = 32.5 m, in the shoaling region, are compared with model results in figures 11, 12, 13 and 14. The model is seen to predict well both surface elevation ζ and velocity distribution u(z).

Finally, the 2D version of the model is applied to study wave propagation over a coplicated geometry used by BERKHOFF *et al.* (1982). This experiment is a standard test for verifying models based on mild-slope equation. Monochromatic waves with period 1 sec and height H = 4.64 cm are generated by a wavemaker at y = − 10 m. The experimental topography consists of an elliptic shoal resting on a plane sloping bottom with a slope 1:50. The plane slope rises from a region of constant depth d = 0.45 m (d/Lo = 0.288, i.e. intermediate water) and the entire slope is turned at an angle of 20° to a straight wave paddle. The slope is described by: d = 0.45 m for $y' < -5.82$ m and $d = 0.45 - 0.02(5.82 + y')$m for $y' > -5.82$ m (x′ and y′ are the slope-oriented coordinates). The boundary of the shoal is given by $(x'/4)^2 + (y'/3)^2 = 1$ and the thickness of the shoal is $d = -0.3 + 0.5(1-(x'/5)^2 - (y'/3.75)^2)^{1/2}$.

Wave heights along eight sections near the shoal were measured in the experiment. Figure 11 shows the comparison of experimental data and model results. The agreement is reasonably good.

Runs of the above three tests without the additional non linear Serre terms, of order $O(\epsilon\sigma^2)$, in the equations (27) and (28) were also made. The agreement was generally good but not as good as seen using these terms. The differences are more significant in the shoaling test by BUHR HANSEN and SVENDSEN (1979) and in the 2DH propagation over the shoal. The need of inclusion additional terms is also pointed out by WEI and KIRBY (1995) and WEI *et al.* (1995).

CONCLUSIONS

A new form of two types of the Boussinesq equations has been presented in this work with corrected dispersion characteristics for monochromatic waves only. Considering nonlinear waves, the new equations are now valid both in intermediate and shallow water keeping a unified form. Using the mean over the depth horizontal velocity as the velocity variable the continuity equation, which is exact to all orders of ϵ and σ^2, remains unchanged. In the momentum equation nonlinear terms of order up to $O(\epsilon\sigma^2)$ are considered.

The distribution of the velocities w, u and pressure p (based on an analytical solution of the Laplace equation) are not similar to those of classical Boussinesq equations as derived in PEREGRINE (1972) and mainly used by many researchers (i.e. parabolic u(z) and p(z)).

The proposed model is tested against analytical solution and experimental data. The wave height in shoaling region, the surface elevation profile and the horizontal velocity distribution are predicted very well. Thus the new model can be used for the simulation of the non-linear wave propagation in intermediate and shallow water.

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