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Representing Equilibrium Beach Profiles with an Exponential Expression

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ABSTRACT

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Representative beach profiles from the U.S. east coast and the Gulf of Mexico were fit to two mathematical expressions below the water line: (1) the well-known $h = Ax^{2/3}$ shape, and (2) the exponential shape $h = B(1 - e^{-kx})$. For a majority of the profiles, the exponential shape more closely approximated the measured data. Generally, the exponential shape may be an improved representation of equilibrium beach profiles—particularly when grain size varies in the offshore direction. Its leading coefficient may yield insight to the selection of a functional depth of closure and its exponential coefficient may be related to sediment size and/or its offshore gradation.

ADDITIONAL INDEX WORDS: *Beach nourishment, beach profile, coastal engineering, equilibrium beach profile.*

INTRODUCTION

A mathematical expression to describe beach profile shape is central to almost every formulation of currents, wave dynamics, sediment transport, and shoreline response across the surf zone. Various expressions proposed or used over the years have ranged from the most simplistic linear relationship to complicated empirical relationships (KEULEGAN and KRUMBEIN, 1919; SWART, 1974; VELLINGA, 1983; SUNAMURA and HORIKAWA, 1974; SUH and DALRYMPLE, 1988; HAYDEN *et al.*, 1975; WINANT *et al.*, 1975; WEISHAR and WOOD, 1983; AUBREY, 1979). Many of these latter expressions are a welcome improvement over the unrealistic assumption of a linear beach profile because the linear profile assumption can significantly bias formulations of surf zone dynamics (see, for example, BODGE, 1988). However, many of these expressions are also quite complicated and cumbersome to apply.

BRUUN (1954) and DEAN (1977, 1991) each proposed an equilibrium beach shape given by

$$h = Ax^m \qquad (1)$$

where h is the still-water depth, x is the horizontal distance from the shoreline, A is a dimensional shape parameter, and $m = 2/3$. The simplicity of this expression lends it great utility.

BRUUN (1954) found that Equation 1 well-approximated beach profiles from the Danish North Sea and Mission Bay, California. DEAN (1977) similarly found that Equation 1 reasonably represented 504 beach profiles collected by HAYDEN *et al.* (1975) along the U.S. east coast and the Gulf of Mexico. Additionally, DEAN (1977) showed that Eq. (1), with $m = 2/3$, is consistent with uniform wave energy dissipation per unit volume across the surf zone. MOORE (1982) and DEAN (1987), respectively, also described a potential relationship between the shape parameter A and grain size, and between A and fall velocity.

EXPONENTIAL FUNCTION

In the course of routine work with beach profiles from the southeast U.S. and the Caribbean, the author often found that an *exponential* expression better approximates equilibrium beach shapes. Specifically, a profile of the form

$$h = B(1 - e^{-kx}) \qquad (2)$$

was considered, where B and k have dimensions of depth and distance^{-1}, respectively.

Equations 1 and 2 were fit to a condensed set of the 504 beach profiles which DEAN (1977) originally used to investigate the $h = Ax^m$ shape. That is, Dean presented ten (10) data groups characterizing ten reaches of the shoreline from New York through Texas (Table 1). An average beach

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profile was developed for each of the ten data groups.

In the present investigation, these ten average profiles were fit by least-squares technique for

(i) Equation 1 with A and m variable,
(ii) Equation 1 with m = ⅔, and
(iii) Equation 2 with B and k variable.

The quality of fit for each was determined by

$$\epsilon = \frac{\Sigma\,(h_i - h_{pi})^2}{\Sigma\,h_i^2} \times 100\% \qquad (3)$$

where h is the "actual" depth and h_p is the depth predicted by either Equation 1 or 2. The subscript i refers to each of 50 uniformly-spaced points along the profile used to describe the profile. From Equation 3, the value $\epsilon = 0$ corresponds to a perfect fit, and increasing values of ϵ refer to increasingly poorer fit.

Profile data were available up to 365 m (1,200 feet) from shore. The preferred vertical datum was Mean Low Water (MLW) but varied within the original data set (DEAN, 1977). Data above the nominal shoreline elevation were not considered here.

RESULTS

Table 2 describes the best-fit values of the coefficients in Eqs. 1 and 2 for each of the ten data groups described above. Figure 1 graphically compares the "actual" and "best-fit" beach profiles for each group.

For six of the ten groups, the exponential expression (Eq. 2) yields an improved approximation of the actual profile relative to the $h = Ax^m$ expression. Of these six, the exponential expression is a *significant* improvement over $h = Ax^m$ in half the cases (*viz.*, data groups I, V, and VIII).

Of the other four groups for which the exponential expression is not a better fit, two represent poorly conditioned profiles which neither expression describes well (*viz.*, data groups IV and VII). Likewise, although Group III is better fit by the exponential expression, it is not particularly well described by either expression. If data groups III, IV, and VII are neglected (because neither expression represents the "actual" profiles particularly well), then the exponential expression is an improved predictor over $h = Ax^m$ for five out of the seven remaining groups (*i.e.*, 71% of the cases).

It is noted that the bulk of the profile data (46%) is included in data group VI. (All of the profiles in this group are from the east coast of Florida.) For this group, the Ax^m expression is an improved predictor over the exponential expression. However, this group (which contains 46% of the profile data) includes less than 15% of the entire study area's coastline. Therefore, the poorer exponential fit for this group is not as significant as it may seem.

The overall, weighted-average quality of fit computed for all ten data groups is about the same for the Ax^m and exponential expressions; *i.e.*, 0.32% and 0.34% respectively. The overall quality of fit of the exponential expression is markedly superior to that of the $Ax^{2/3}$ expression, *i.e.*, 0.34% *vs.* 0.75%. Of course, one expects that a 2-parameter model (*i.e.*, the exponential expression) should yield a better fit than the $Ax^{2/3}$ model, which has only one free parameter.

In routine use of $h = Ax^m$, a value of ⅔ is taken for m. Table 2 indicates that this simplification is reasonably appropriate, relative to a variable m, for perhaps half the data groups (*viz.*, II, VI, VII, IX, and perhaps X). The quality-of-fit for m = ⅔ is best for data group VI (along Florida's east coast). However, because this group includes 46% of the data but less than 15% of the study area's coastline, it has a disproportionate weight to the study's results. (That is, the recommended value of m = ⅔ is disproportionately based upon profiles concentrated along Florida's east coast.)

The quality of the exponential expression is superior to the $Ax^{2/3}$ expression for eight of the ten data groups and is about the same for a ninth. If data groups III, IV, and VII are again neglected, then the exponential expression yields an improved fit over $h = Ax^{2/3}$ in 86% of the remaining cases.

With the exception of data group VI, the improved quality of the exponential expression's fit is most apparent near the shoreline. In this area, the $h = Ax^m$ shape generally over-predicts profile curvature and depth.

For the ten data groups tested, there is no obvious geographic location nor profile trait to indicate which profiles are better fit by the Ax^m expression rather than by the exponential expression. As expected, profiles which exhibit terrace or bar/trough features are fit relatively poorly by both expressions.

DISCUSSION

The physical significance of the exponential expression is interesting. The leading coefficient, B, is a depth which is asymptotically reached off-

Table 1. *Beach profile data groups (after DEAN, 1977).*

Data Group	Profiles	Locations: From	Locations: To
I	1–35	Montauk Point, NY	Rockaway Beach, NY
II	36–78	Sandy Hook, NJ	Cape May, NJ
III	79–116	Fenwick Light, DE	Ocean City Inlet, MD
IV	117–145	Virginia Beach, VA	Ocracoke, NC
V	145–159	Folly Beach, SC	Tybee Island, GA
VI	160–394	Nassau Sound, FL	Golden Beach, FL
VII	395–404	Key West, FL	Key West, FL
VIII	405–439	Caxambas Pass, FL	Clearwater Beach, FL
IX	440–477	St. Andrew Pt., FL	Rollover Fish Pass, TX
X	478–504	Galveston, TX	Brazon Santiago, TX

shore. While this is intuitively troubling in the grand scale (because we know that depths eventually increase towards the ocean floor), it may be of value when working nearshore. That is, beach profile representations are generally intended for nearshore (surf zone) application. Defining the offshore limit of the nearshore zone is important and usually troublesome. For example, this limit is often required to describe the offshore limit of sediment transport or the depth to which a beach nourishment project will equilibrate (*i.e.*, the "depth of closure"). In those cases where the exponential expression exhibits a reasonable fit to a profile, its leading coefficient B may yield valuable insight to selection of a functional depth of closure.

The exponential term k in Eq. 2 describes profile curvature. One might suspect that while the leading coefficient B is determined by wave and sediment characteristics, the k term might be significantly influenced by sediment characteristics and particularly by sediment gradation across the surf zone.

Adoption of a fixed value of k in Eq. 2 is ill-advised. For the ten data groups examined, k ranges from 3×10^{-5} m^{-1} (0.0001 ft^{-1}) to 1.16×10^{-3} m^{-1} (0.0038 ft^{-1}). The mean and standard deviation, respectively, are 5.12×10^{-4} m^{-1} (0.00168 ft^{-1}) and 4.1×10^{-4} m^{-1} (0.00134 ft^{-1}). There is considerably less variation for m in Eq. 1, which ranges from 0.385 to 0.914, with mean and standard deviation of 0.64 and 0.194, respectively.

No attempt is made herein to relate the parameters k and B to beach conditions. While the ability to predict these parameters is of ultimate importance, they should at present be selected locally from profile analysis. In practice, this is also what

Table 2. *Parameters and quality for $h = Ax^m$ and $h = B(1 - e^{-kx})$ fitting of characteristic beach profile groups along the U.S. East Coast and Gulf of Mexico.*

Data Group	Best-Fit Parameters: A ($m = 2/3$) $[ft^{1/3}]$	A $[ft^{1/3}]$	m	B [ft]	k $[ft^{-1}]$	Quality of Fit (%): $Ax^{2/3}$	Ax^m	$B(1 - e^{-kx})$	N	No. Profiles: % of Total
I	0.159	1.035	0.385	14.49	0.00384	2.94	0.83	0.20	35	6.9
II	0.217	0.167	0.706	31.85	0.00115	0.26	0.23	0.03	43	8.5
III	0.189	0.036	0.914	77.23	0.00030	1.45	0.52	0.48	38	7.5
IV	0.182	0.037	0.905	199.43	0.00010	1.43	0.67	0.74*	29	5.8
V	0.091	0.474	0.419	8.59	0.00331	2.44	0.84	0.10	15	3.0
VI	0.156	0.214	0.620	20.59	0.00138	0.09	0.05	0.37**	234	46.4
VII	0.056	0.022	0.803	21.36	0.00032	1.62	1.38	1.66**	10	2.0
VIII	0.125	0.591	0.433	11.92	0.00318	2.18	0.77	0.07	35	6.9
IX	0.093	0.118	0.632	11.99	0.00145	0.47	0.45	0.32	38	7.5
X	0.099	0.181	0.578	11.70	0.00174	0.22	0.06	0.20*	27	4.4
				Weighted Average		0.76	0.32	0.34		

*Exponential profile fit poorer than Ax^m

**Exponential profile fit poorer than Ax^m and $Ax^{2/3}$

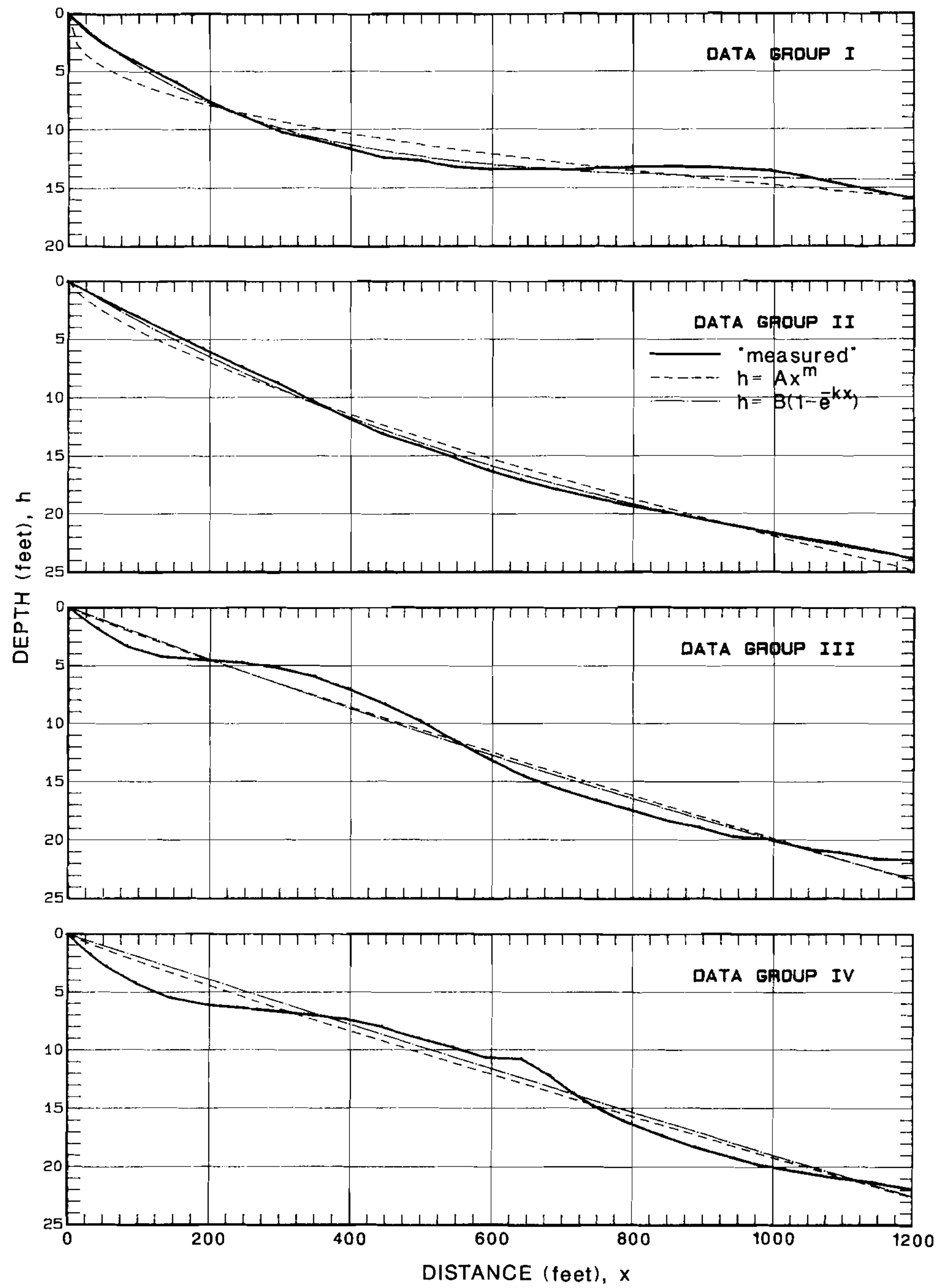

Figure 1. Comparison of "measured" profiles and the expressions $h = Ax^m$ and $h = B(1 - e^{-kx})$. (Continued)

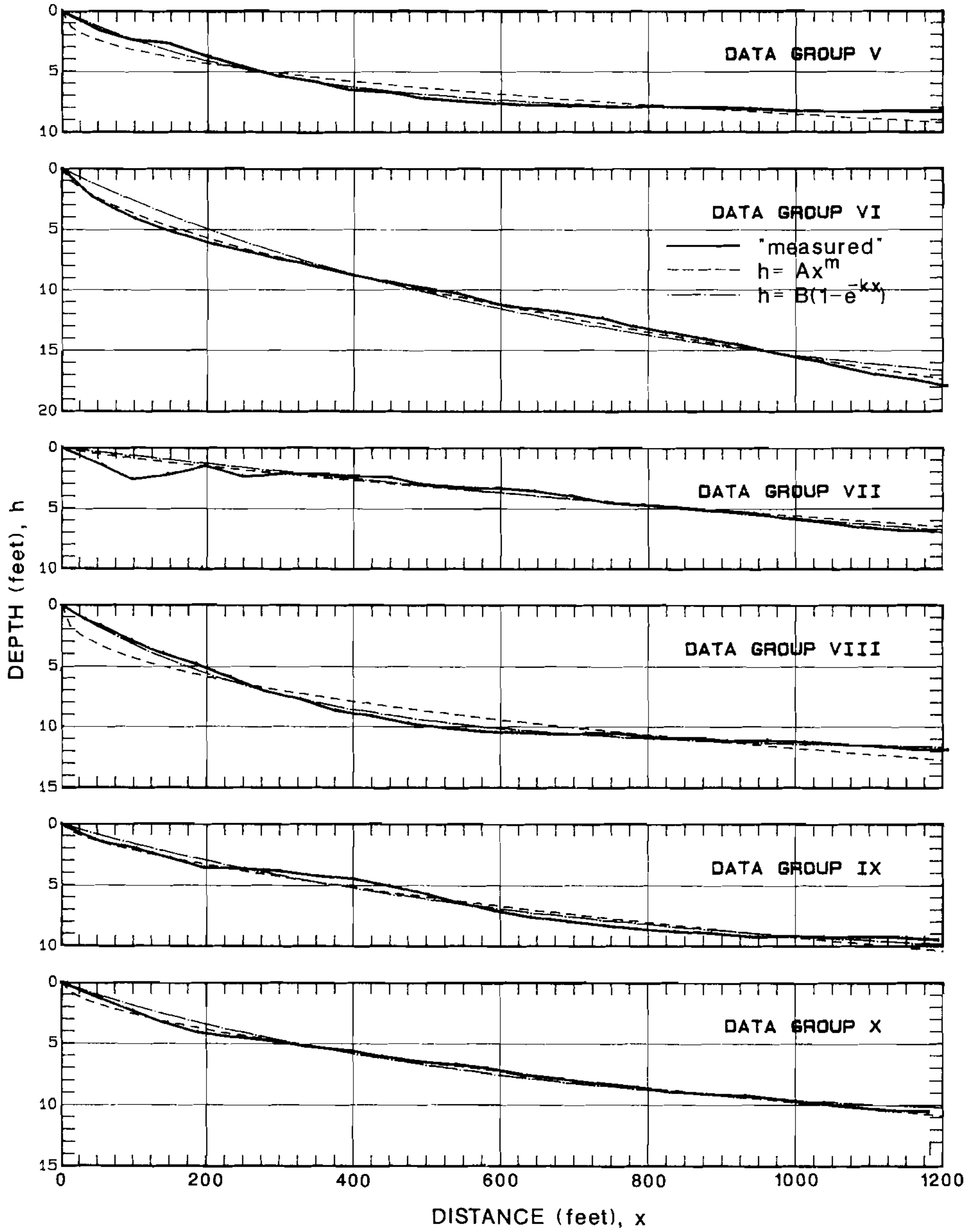

Figure 1. (Continued.)

is generally done to select the A parameter in $Ax^{2/3}$ (and m in Ax^m). There is, at present, insufficient confidence in suggested relationships between the A parameter and sediment size or fall velocity for reliable, site-specific prediction of the A parameter.

In any event, this paper's purpose is not to *predict* the actual beach shape parameters; its intent is to present an *alternate expression* for the beach shape itself. Such expressions are useful of themselves in theoretical investigations of cross-shore phenomena. They are useful in site-specific application for characterizing similar measured beach profiles.

Table 3. *Parameters and quality of $h = Ax^m$ and $h = B(1 - e^{-kx})$ fitting of hypothetical beach profiles with decreasing offshore grain size.*

	Best-Fit Parameters					Quality of Fit (%)		
Figure	A (m = ⅔) [$ft^{1/3}$]	A [$ft^{1/3}$]	m	B [ft]	k [ft^{-1}]	$Ax^{2/3}$	Ax^m	$B(1 - e^{-kx})$
2a	0.271	0.497	0.573	27.92	0.00213	0.211	0.030	0.098
2b	0.194	1.591	0.340	15.20	0.00532	3.397	0.462	0.028
2c	0.199	1.362	0.369	15.91	0.00474	2.594	0.274	0.108

It may also be argued that $h = Ax^{2/3}$ is a more rational equilibrium profile expression, because it supports the notion of uniform energy dissipation per unit volume across the surf zone (DEAN, 1977). However, note that this was developed under three important considerations: (1) linear wave theory applies, (2) local wave height is a fixed proportion of local water depth, and (3) the dissipative quality of the bed is uniform across the surf zone. Simplistically, the latter implies that sediment characteristics are uniform across shore. This is often not the case in nature. Real beaches often exhibit coarse sands near the shoreline and increasingly finer sands offshore.

Generally, the improved quality of the exponential expression's fit may be most notable for profiles with varying offshore grain size. Figure 2 illustrates this effect for three hypothetical cases. In each, an $h = Ax^{2/3}$ profile was fabricated, but the value of the A parameter was discretely decreased in the offshore direction to simulate increasingly smaller sand size. Specifically,

$$h(x) = (A_{n-1}x_n^{2/3} - A_nx_n^{2/3}) + A_nx^{2/3} \quad \text{for } x_n < x < x_{n+1} \tag{4}$$

where A_n is the shape parameter at each of N sections of the profile bounded by x_n to x_{n+1}. (The leading term in parenthesis ensures continuity of depth between sections.)

In Figure 2a, the A value in each section was cumulatively decreased by 10% of the shorewardmost A value at 61-m (200-ft) intervals; *i.e.*,

$$A_n = A_o(1 - 0.1n) \tag{5}$$

for $n \geq 0$. In this case, the $h = Ax^m$ profile shape exhibits slightly better fit than the exponential expression (particularly near-shore) (see Table 3). This is not completely surprising since Eq. 5 simply represents a superposition of many Ax^m shapes where all A values are linearly related. However, it is noted that the exponential expression yields a significantly improved fit over the $h = Ax^{2/3}$ shape—that is, where m is fixed as ⅔. (The $h = Ax^{2/3}$ fit is not shown in the figures.)

In Figure 2b, the A parameter in each section is reduced to 50% of its adjacent shoreward section at 61-m (200-ft) intervals; *i.e.*,

$$A_n = 0.5A_{n-1} = (0.5)^nA_o \tag{6}$$

Here, the quality of the exponential fit improves over the previous case, while that of the Ax^m fit degrades. In fact, the exponential expression represents a significant improvement over the $h = Ax^m$ expression and a very significant improvement over $h = Ax^{2/3}$ (see Table 3). Unlike the previous case, Eq. 6 is a superposition of many Ax^m shapes for which the A values are *not* linearly related.

The hypothetical profile in Figure 2c is identical to that of Figure 2b, except that the A value is decreased only at x = 61 m and x = 122 m (x = 200 ft and x = 400 ft). (That is, the sand and profile shape characteristics are assumed to be uniform for $x \geq 122$ m.) This is likely more representative of nature than the profile shown in Figure 2b. In this case, the exponential expression is still an improvement over an Ax^m expression, although not as significantly as in Figure 2b. In each of Figures 2a through 2c (and Eqs. 5 and 6), $A_o = 0.3$.

These results are consistent with findings released while the present paper was in review. Using "blindfolded" tests, WORK and DEAN (1991) found that an "A" parameter which varies exponentially offshore (to simulate decreasing grain size) produced a better quality of profile-fit in 3 out of the 4 cases tested than did an Ax^m profile with fixed A or linearly-varying A. The fourth case produced similar fit quality for all three profile-types.

Likewise, LARSON (1991) described improved quality of fit for three field profiles by considering

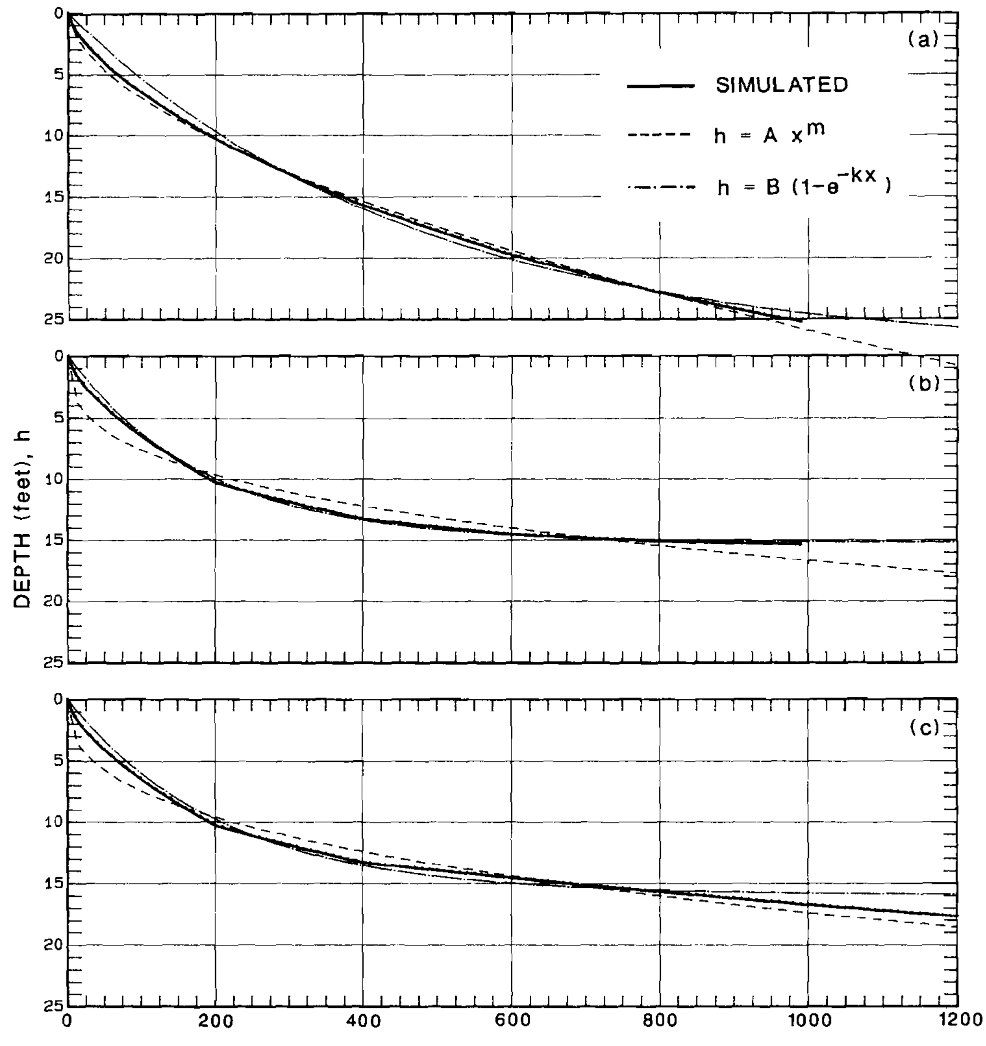

Figure 2. Hypothetical profiles with decreasing offshore grain size (bold) fit to the expressions $h = Ax^m$ and $h = B(1 - e^{-kx})$. The latter expressions are indicated by dashed lines.

an A parameter which varies with the local equilibrium energy dissipation where the latter is assumed to exponentially decrease in the offshore direction. LARSON demonstrates that such a decrease is consistent with measured grain size variations for the three beaches.

SUMMARY

For the ten self-similar data groups which describe the 504 profiles used by DEAN (1977) to test the $h = Ax^{2/3}$ profile shape, the majority (60% to 71%) were better fit by an exponential expression (Eq. 2) relative to the $h = Ax^m$ shape. Similarly, 80% to 86% were better fit by the exponential expression (Eq. 2) relative to the $h = Ax^m$ shape where $m = 2/3$. The average quality of fit (weighted by the number of profiles in each group) was similar for the Ax^m and exponential expressions, but was superior by a factor of two for the exponential expression relative to an $Ax^{2/3}$ expression. The data group with the largest number of profiles (46% of the total) was better fit by the Ax^m and $Ax^{2/3}$

expressions; however, this group describes less than 15% of the study area's coastline.

In those cases for which the exponential expression was superior to $h = Ax^m$, the improvement in quality-of-fit was significant. (Within the $h = Ax^m$ group, it is also noted that a fixed value of $m = 2/3$ yields a reasonably accurate fit for about half the data groups tested, and the great majority of these are concentrated along Florida's east coast, or less than 15% of the study area.)

The improved fit of the exponential expression is likely to be most notable for beaches with decreasing grain size (or fall velocity) in the offshore direction. In contrast to the $h = Ax^m$ profile shape, which sometimes unrealistically describes constantly increasing depth offshore, the exponential expression implies that the beach asymptotically approaches a *fixed* depth offshore. However, because both expressions are intended for nearshore applications in the first place, this shortcoming might be neglected far outside the surf zone. In this way, it is conjectured that the leading coefficient of the exponential expression may yield insight to the depth of closure. The coefficient within the exponential argument may describe sediment type or its gradation in the offshore direction.

No attempt is made to correlate the exponential expression's shape parameters with beach or surf characteristics. Instead, the paper is intended to introduce an alternate expression for the beach shape which may be of use to theoretical investigations of cross-shore processes, or for characterizing profile shape when site-specific data are available.

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□ RÉSUMÉ □

Plusieurs profils de plage sous-marine représentatifs de la côte est des Etats Unis et du Golfe du Mexique ont été ajustés à deux expressions mathématiques: (1) la forme bien connue $h = Ax^{2/3}$ et (2) la forme exponentielle $h = B(1 - e^{-kx})$. Pour la plupart des profils, la forme exponentielle fournit une bonne représentation du profil d'équilibre, en particulier quand la taille du sédiment varie avec l'éloignement du rivage. Le premier coefficient peut conduire à la sélection d'une profondeur de clôture, le coefficient exponentiel peut être rapporté à la taille du sédiment et/ou à sa variation en fonction de l'éloignement vers le large.—*Catherine Bousquet-Bressolier, Géomorphologie EPHE, Montrouge, France.*

□ RESUMEN □

Se ajustó por debajo del nivel medio del agua a dos expresiones matemáticas perfiles de playa representativos de la costa Este de Estados Unidos y del Golfo de México: (1) La bien conocida forma $h = Ax^{2/3}$ y (2) la forma exponencial $h = B(1 - e^{-kx})$. Para la mayoría de los perfiles, la forma exponencial se aproximaba más estrechamente a los datos medidos. Generalmente, la forma exponencial puede ser una representación mejorada de los perfiles de equilibrio, particularmente cuando el tamaño de grano varía en la dirección perpendicular a la playa. Su coeficiente director puede llevar a discernir la selección de la profundidad funcional de cierre y su coeficiente exponencial puede relacionarse con el tamaño de sedimentos y/o su gradación a lo largo de él.—*Department of Water Sciences, University of Cantabria, Santander, Spain.*