Reaching New Heights with Ambiguous Rockets

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The height of a rocket is modeled with respect to time by taking into account its thrust, the rate at which its mass changes, and assuming that it flies straight up. Several simplifying assumptions are made, namely; thrust, force due to gravity and the change in mass with respect to time are all constant, and that drag is negligible.

Nomenclature

Limit aperiodic groups and G-sets

1 Introduction

Rockets are complicated dynamic systems that can be difficult (if not overwhelming) to understand. However, with much of modern life relying on these rockets mainly because of the satellites they put into orbit, this understanding is critical. Perhaps the best way to start building this understanding of rockets is by looking back at their history.

Although accounts of simple, firework-like rockets can be traced back with some controversy to the Chinese c. 1250 A.D., modern rockets, as we know them, first started to be developed by Robert Goddard in 1912. He developed many concepts including: staging which is burning the fuel in smaller combustion chambers, (instead of having just one large tube of solid fuel). The use of the de Laval nozzle [1] to allow the propellant accelerate out at supersonic speeds. He also toyed with gyroscopic guidance systems and regenerative cooling (the process of wrapping the fuel lines around the nozzle to keep it cool while firing). After Goddard, all of these concepts were developed and expanded upon throughout WWII and the cold war. With the race to the moon, computers were integrated and advanced crafts capable of maintaining livable conditions were developed. [2] Once all these concepts were combined, the pieces of the modern rocket were in place.

However, this brief history only gives an overview of the names and technologies that go into a rocket, providing a rough understanding of what is involved. Digging a little deeper into the physics behind any one part can help expand it a little more.

Starting with the "Ideal Rocket Equation" [3]

$$\Delta v = I_{sp} * g_0 * ln\left(\frac{m_f}{m_i}\right),$$

where I_{sp} is the Specific impulse, g_0 is acceleration due to gravity and m_f and m_i stand for the initial and final masses of the rocket, also known as the Tsiolkovsky rocket equation, yields the change in velocity (Δv) of a rocket. When trying to determine how far a rocket can go its Δv value is like the gas tank in a car: if it has a small value then it cannot go very far.

Now when in the vacuum of space most of the variables remain constant, but when one considers a rocket ascending through the atmosphere one can begin to realize just how dynamically difficult the problem is. Looking at just one of the variables in this equation and figuring out how it relates to atmospheric pressure can help one get a sense of how involved this really is.

I_{sp} (or specific impulse) when multiplied with g_0 yields effective exhaust velocity v_e, an indicator of how powerful a rocket is. The equation for specific impulse is [1]:

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$$I_{sp} = \frac{F}{\dot{m} * g_0},$$

where the force F , and $\dot{m}$ is the mass flow rate. F (force) has its own equation [1]

$$F = \dot{m} * v_e + A_e(p_1 - P_2),$$

where v_e is effective exhaust velocity, A_e is the area of the opening at the tail end of the rocket bell, p_1 is the static pressure at that opening and P_2 is the atmospheric pressure.

The above equation finally gets to P_2 (atmospheric pressure) which can be modeled as [1]

$$P_2 = p_0 * e^{\left(\frac{h}{h_0}\right)},$$

It is a relatively simple exponential function, where p_0 is pressure at sea level, h_0 is the height of sea level and the whole thing is dependent on the instantaneous height h.

This is just the tip of the iceberg. The above deconstruction only followed one path: picking one variable out of each equation to trace back to its solution. As each variable was picked it left behind others whose origins would be similarly complicated. Furthermore, many convoluting factors such as drag and thermal effects were left out. Otherwise the analysis would be hopelessly sophisticated and of little illustrative use. This is why in the title we used the phrase "Ambiguous Rockets." What we are detailing is essentially the simplest, most generic mathematics behind any possible rocket.

1.1 Problem Statement

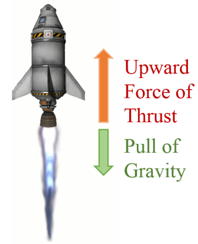
Originally, rocket-engine efficiency was to be modeled with respect to atmospheric pressure. The vision was to plot efficiency over the course of a typical accent profile. However this would have been beyond the scope of this project. Therefore the model shifted to rocket height, going straight up, in terms of time: the equation was derived by taking into account thrust, the mass lost by expelling fuel and the opposing force of gravity.

2 Methods & Materials

Fig.1 A basic diagram of the model.

Newton's 2nd Law: Force = mass * acceleration

$$\sum F = m \cdot a$$



$$F_{Thrust} - F_{gravity} = m \cdot a$$

Initial conditions: Since the rocket begins at rest at height 0, the initial conditions are as follows:

$$1. h(0) = 0$$
$$2. h'(0) = 0$$
$$3. h''(0) \geq 0$$

Assumptions: In order to make the project more comprehensible several simplifying assumptions are made:

1. $F_{Thrust} = -k\frac{dm}{dt}$ [1], where k is a constant. The "$-$" (negative sign) indicates that the force moves opposite to the movement of the gases.
2. $\frac{dm}{dt}$ is a constant c where c is negative.
3. $F_{gravity} = -m \cdot g$, where g is constant close to earth.
4. The rocket moves in a straight vertical path making the position equal to the rockets height at any point in time. Therefore $a = h''$

Combining these assumptions results in:

$$-k\frac{dm}{dt} - m(t)g = m(t) \cdot h''(t)$$
$$-\frac{k}{m(t)}c - g = h''(t)$$

The function $m(t)$ represents the mass of the rocket as a function of time and can be written as $m(t) = m_o - c \cdot t$. Therefore the equation becomes:

$$h''(t) = -\frac{k}{m_o + ct} \cdot c - g$$

Note how the second term of the function $m(t)$ is now positive since $c > 0$.
Now there is an important restriction to be made before moving on. At $t = 0$, h'' becomes:

$$h''(t) = -\frac{kc}{m_o} - g \geq 0 \tag{1}$$

This is because at $t = 0$ the acceleration provided by the thrust needs to be greater than or equal to the acceleration due to gravity otherwise the rocket would accelerate downward as if the ground were not there to stop it.

Solving the second degree equation:

$$
\begin{aligned}
-\int h''(t)dt &= \int \left(\frac{k}{m_o + ct} \cdot c + g\right) dt \\
-h'(t) &= \int \left(\frac{k}{m_o + ct} \cdot c + g\right) dt
\end{aligned}
$$

Solve for $h'(t)$:

$$
\begin{aligned}
-h'(t) &= \int \left(\frac{k}{m_o + ct} c + g\right) dt \\
&= \int \left(\frac{k}{m_o + ct}\, c\right) dt + \int g\, dt
\end{aligned}
$$

Set $u = m_o - ct,\ du = -c\, dt,\ \frac{du}{-c} = dt$. Then:

$$
\begin{aligned}
-h'(t) &= \left[\int \left(\frac{kc}{u}\right) \frac{du}{-c}\right] + gt \\
&= \left[k \int \left(\frac{1}{u}\right) du\right] + gt \\
&= k \cdot \ln u + gt + a \\
&= k \cdot \ln m_o - ct + gt + a
\end{aligned}
$$

Solving for the constant a'' using initial conditions at $t = 0$ yields a second restriction (2):

$$
\begin{aligned}
-h'(0) &= k \cdot \ln(m_o - (c \cdot 0)) + (g \cdot 0) + a \\
0 &= k \cdot \ln(m_o) + a
\end{aligned}
$$

$$
a = -k \cdot \ln(m_o) \tag{2}
$$

Similarly, we solve for $h(t)$ to get:

$$
\begin{aligned}
-h'(t) &= -k \ln(m_o + ct) + gt + a \\
-\int h'(t)dt &= \int (-k \ln(m_o + ct) + gt + a) dt \\
-h(t) &= \int -k \ln(m_o + ct) dt + \int gt\, dt + \int a\, dt
\end{aligned}
$$

Set $u = m_o - c \cdot t,\ du = -c\, dt,\ \frac{du}{-c} = dt$

$$
-h(t) = \int -k \ln u \frac{du}{-c} + \int gt\, dt + \int a\, dt
$$

$$
-h(t) = \left[\frac{k}{c} \int \ln u\, du\right] + \frac{g}{2} t^2 + at \tag{3}
$$

Next, we apply the integration by parts method to evaluate the integral:

$$
\begin{aligned}
\int (u)'(\ln u) du &= (u)(\ln u) - \int (u)(\ln u)' du \\
\int 1 \cdot \ln u\, du &= (u)(\ln u) - \int (u)(\frac{1}{u}) du \\
\int \ln u\, du &= (u)(\ln u) - u
\end{aligned}
$$

Plug in (3):

$$
\begin{aligned}
-h(t) &= \frac{k}{c}[(u)(\ln u) - u] + \frac{g}{2} t^2 + at + b \\
&= \frac{k}{c}[(m_o + ct)(\ln(m_o) + ct) - (m_o + ct)] + \frac{g}{2} t^2 + at + b
\end{aligned}
$$

Next, we find the constant b'' using the initial condition. This will result in the third restriction (4):

$$
\begin{aligned}
-h(0) &= \frac{k}{c}[(m_o - (c \cdot 0)(\ln(m_o) - (c \cdot 0)) \\
&\quad - (m_o - (c \cdot 0)] + \frac{g}{2}(0)^2 + (a \cdot 0) + b \\
0 &= \frac{k}{c}[m_o(\ln(m_o)) - m_o] + b
\end{aligned}
$$

$$
b = \frac{k}{c} m_o [1 - (\ln(m_o))] \tag{4}
$$

Therefore:

$$
\begin{aligned}
-h(t) &= \frac{k}{c}[(m_o + ct)(\ln(m_o) + ct) - (m_o + ct)] \\
&\quad + \frac{g}{2} t^2 - (k \cdot \ln(m_o))t + \frac{k}{c} m_o [1 - (\ln(m_o))],
\end{aligned}
$$

Simplifying yields the solution:

$$
\begin{aligned}
h(t) = & -\frac{k}{c} m_o \cdot \ln(m_o + ct) - kt \cdot \ln(m_o + ct) \\
& + kt - \frac{g}{2} t^2 + kt \cdot \ln(m_o) + \frac{k}{c} \ln(m_o)
\end{aligned}
$$

The range of time over which the rockets height can be modeled is limited by mass. That is, once the rocket runs out of mass to expel thrust is no longer constant. Therefore:

$$t \leq -\frac{m_o}{c} \tag{5}$$

3 Results

The solution can be modeled by plotting it in Matlab©. The values are chosen so that all four restrictions are satisfied. These specific values were chosen arbitrarily and fit the required assumptions and conditions laid out above.

$$
\begin{aligned}
k &= 60 \\
c &= -5 \\
m_o &= 25 \\
g &= 9.81
\end{aligned}
$$

Note that all values are unit-less. This is for demonstration purposes, but any set of variables with agreeing units could be used. Plotting the function with these variables generates the following model of rocket height with respect to time. Here, it can be seen what would be expected of a rocket: it lifts off slowly, but as it burns fuel the pull of gravity decreases and the thrust accelerates the rocket upward faster and faster. If the first restriction (1) is invalidated and the thrust was insufficient to counteract gravity ($k = 30$), the result is:

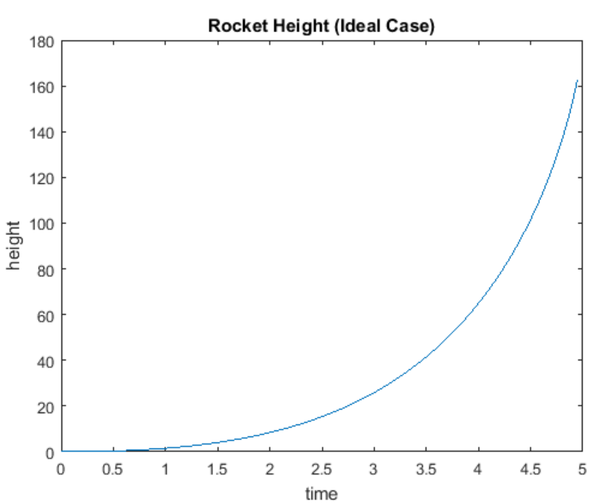


Fig. 2 Rocket Height over an ideal accent.

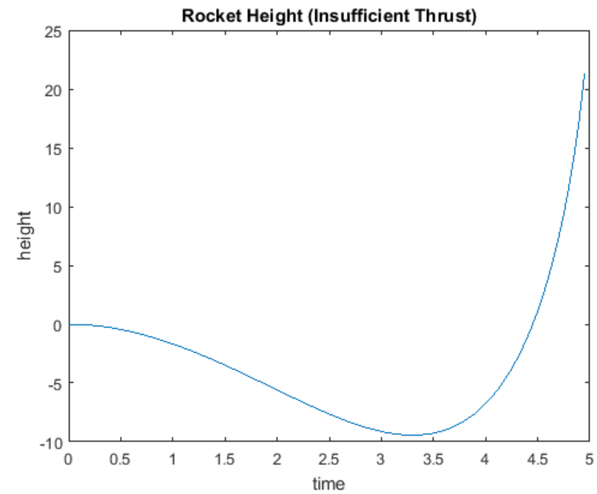


Fig. 3 Rocket Height with weak thrust.

The force of gravity is initially stronger than that of thrust and the rocket actually starts falling below its initial height as if there were no ground to stop it. After about 3.3 units of time the rocket has become light enough to ascend under its own power.

Furthermore, observe how the model behaves after the rocket runs out of mass to expel. Returning the value of thrust back to where $k = 60$ and extending the time-frame to $t = 10$ yields the following graph:

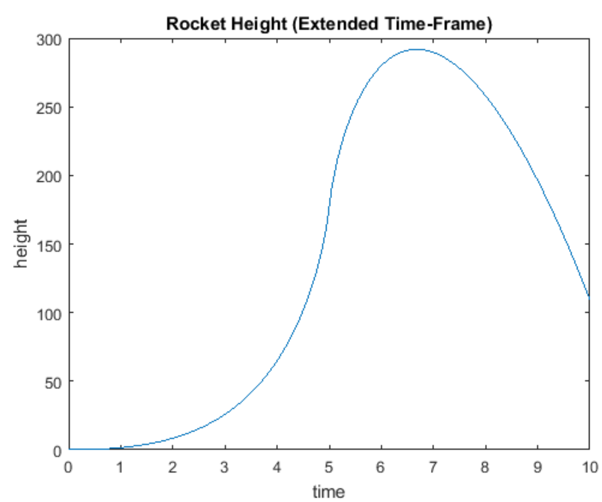


Fig. 4 Rocket Height after all fuel is expelled.

It is important to note that because the model was developed under the conditions of constant thrust and a constant rate of mass loss, it is no longer accurate once $t > \frac{m_0}{c}$ (5). However, the model still shows the general behavior that would be expected of the situation. When the it runs out of fuel at $t = 5$, the inflection of the graph changes. Thrust is no longer being supplied, so gravity is free to pull it back down. The rocket coasts up to just bellow 300 units of height before gravity cancels out its momentum, and it falls from the sky.

4 Conclusions

The solution proved capable of modeling rocket height over time. Given that all variables meet the four restrictions, the model will show the rocket flying upward with increasing acceleration. Likewise, the first restriction was justified by choosing variables that invalidate it: the plot showed the expected behavior of initially falling. Remarkably, the solution could even predict the general behavior of the rocket after it had run out of fuel, though perhaps not accurately. This model could be used as a basic check to see if a rocket will fly before any resources are put in to actually building one. Ultimately it succeeded in demonstrating rocketry's most basic principle: thrust.

Acknowledgements

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References

[1] Huzel, D. K., and Huang, D. H., 1992. "Modern engineering for design of liquid-propellant rocket engines". *AIAA,* **1**(5), May, pp. 1–3.

[2] Ryan, R., 1996. "A history of aerospace problems, their solutions, their lessons". *NASA*.

[3] Ullah, Rizwan, D.-Q. Z. P. Z. M. H., and Sohail, M. A., 2013. "An approach for space launch vehicle conceptual design and multi-attribute evaluation". *Aerospace Science And Technology*.

Appendix A: Matlab Code

```
%Variables
k=60;
c=-5;
m=25;
g=9.81;
t = linspace(0,5,100);

%The solution
h = (-(k/c)*m*(log(m+c*t)))-(k*t.*(log(m+c*t)))+
(k*t)-((g/2)*(t.^2))+
(k*t.*(log(m)))+((k/c).*m.*(log(m)))

%Graphing the solution
plot(t,h)
xlabel('time') % x-axis label
ylabel('height') % y-axis label
title('Rocket Height')
```