

# SIMPLIFY UNCERTAINTY ANALYSIS WITH EXCEL MACROS

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Estimating and reporting reliability in experiments and calculations is an important part of engineering design and analysis. Reporting results from calculations and experiments without some estimation of reliability may invalidate our results. To illustrate, if we report a volume from designing a chemical reactor without taking into account the uncertainty in the design parameters, we risk under sizing a cooling system, which can have catastrophic consequences for exothermic runaway reactions. One measure of reliability comes from uncertainty analysis. Chemical engineering students may learn simple concepts of experimental error and uncertainty analysis in physics and chemistry labs. Their first impressions and experiences with uncertainty are not typically positive. In some cases, this is their first exposure to statistics. Students find the process tedious, labor intensive, and sometimes irrelevant in the context of their limited science and engineering experience. When we bring up the topic of uncertainty analysis in our engineering instructional labs, students groan in anticipation of the laborious, monotonous calculations.

To reinforce the principles of uncertainty analysis and provide students with tools for uncertainty calculations that help to alleviate their anxiety, we have incorporated uncertainty analysis earlier in our program in a one-semester required course on computational methods for engineering problem solving. Our students typically take this course in the second year of our program, about midway through their chemistry, physics, and engineering lab sequences. Prior to our computational methods course, students receive a basic introduction to descriptive statistics and uncertainty in their chemistry and physics courses. Most students in the course have also completed another course on statistical design of engineering experiments.

In our computational methods course, we introduce students to concepts of random and systematic uncertainty in measurements, degrees of freedom, propagation of uncertainty, and expanded uncertainty (confidence intervals). We outline steps for uncertainty analysis and develop spreadsheet tools to simplify the implementation. We use a simple, hands-on classroom demonstration to generate experimental data and help students experience the differences between uncertainties in analog and digital measuring instruments. The exercise involves calculating the density of an object from replicated measurements of dimensions and weight. The students first perform the steps of uncertainty analysis in an Excel worksheet to experience the calculations “by hand.” In a follow-up class exercise, students create an Excel macro that calculates the expanded propagation of uncertainty according to the conventional Guidelines for Analysis of Uncertainty in Measurements (GUM).<sup>[1]</sup> The macro incorporates basic programming methods of loops, logical statements, input and output, user functions, and subroutines. Students finish the course with a deeper understanding and appreciation of their responsibility for reporting reliability of results in terms of uncertainty. They also move on to other courses equipped with tools for simplifying the implementation of uncertainty analysis for most situations they will encounter in their undergraduate experience.

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## UNCERTAINTY ANALYSIS

In an interactive lecture, we present the well-known principles of uncertainty analysis.<sup>[1-4]</sup> We also discuss case studies to promote the value of uncertainty analysis, including a qualitative example of a nuclear reactor.<sup>[5]</sup> For brevity in this paper, we skip theories of uncertainty analysis that are readily available in the literature and outline the steps for students to follow when calculating the propagation of uncertainty through their engineering calculations. The reader is encouraged to review the literature for background information on uncertainty analysis.<sup>[3]</sup> The analysis presented here is limited by the assumptions of normally distributed random errors in replicated measurements with no bias or correlation between independent variables. We also assume that systematic errors are uniformly distributed between the limits of precision in the measurement. Students learn to reduce bias and correlation between independent variables by careful calibration of instruments and randomization of experiments.

Eq. (1) represents the functional relationship between inputs,  $x$  (e.g., experimental measurements), and the output,  $y$ :

$$y = f(x_1, x_2, \dots) \quad (1)$$

The function,  $f$ , represents any combination of calculation steps that incorporate the input values from experimental measurements  $x$  needed for obtaining  $y$ . The question we answer with uncertainty analysis is how the uncertainties in the input variables propagate through the function to give an uncertainty in the output. Ultimately, we want to know  $y$  with some degree of confidence:

$$y = f(x_1, x_2, \dots) \pm U_y \quad (2)$$

where  $U_y$  represents the expanded uncertainty, or confidence interval for the calculated value of  $y$ .

Uncertainty in the values of the input variables comes from a variety of sources that may include measurement mistakes, systematic errors due to our inability to take precise measurements, and random errors that are the results of our inability to control the environment of the experiment.<sup>[1,4]</sup> We use descriptive statistics of mean and standard deviation as the basic building blocks for predicting the value of replicated measurements and their corresponding uncertainty from random errors. We recommend a conservative calculation for the uncertainty from systematic errors according to the precision of the instruments of measurement. Unfortunately, there is no good way to quantify unknown mistakes in uncertainty analysis. We should take steps in our experimental design and data collection to minimize the possibility for mistakes. For our analysis, we follow the Guidelines for the Analysis of Uncertainty in Measurements.<sup>[1]</sup> The following steps of uncertainty analysis are illustrated with a simple hands-on classroom exercise:

1. Begin with a model of the uncertainty in the experimental inputs from the combined effects of random and systematic errors:

$$x = \bar{x} \pm u_x \quad (3)$$

where  $\bar{x}$  is the expected value for  $x$  calculated from the mean value of  $n$  replicated measurements:

$$\bar{x} = \frac{1}{n} \sum x \quad (4)$$

and  $u_x$  is the combined uncertainty due to random ( $u_R$ ) and systematic ( $u_Z$ ) errors in the measured values:

$$u_x = \sqrt{u_R^2 + u_Z^2} \quad (5)$$

2. Assume random errors are normally distributed and use the standard error in Eq. (6) for the random uncertainty in terms of the standard deviation  $s$ , and the sample size,  $n$ :

$$u_R = \frac{s}{\sqrt{n}} \quad (6)$$

Students discover that they may reduce the random uncertainty in Eq. (6) by taking more measurements. However, this revelation is tempered by Eq. (5) when they realize that systematic uncertainties are always present for any measurement due to the limitations of precision in our instrumentation.

3. Assume that systematic (fixed) errors are uniformly distributed between the limits of readability of the instrument of measurement,  $\pm\delta$ , for a probability density function  $p = 1/(2\delta)$  when  $\bar{x} - \delta \leq x \leq \bar{x} + \delta$ , and  $p = 0$  otherwise. Calculate the variance for the fixed uncertainty in the measurement:

$$u_Z^2 = \int_{\bar{x}-\delta}^{\bar{x}+\delta} \frac{(x-\bar{x})^2}{2\delta} = \frac{\delta^2}{3} \quad (7)$$

or

$$u_Z = \frac{\delta}{\sqrt{3}} \quad (8)$$

Eq. (7) represents a conservative estimate for the fixed error by assuming an equal probability for error over the range of readability of the instrument.

4. Use the law of propagation of uncertainty (ignoring parameter correlation) for calculating the standard uncertainty in the output in terms of the combined uncertainties of the inputs calculated from Eq. (5):

$$u_y = \sqrt{\sum (cu_x)^2} \quad (9)$$

where  $u_y$  is the standard uncertainty in the output  $y$  from Eq. (1). The parameter  $c$  is the sensitivity coefficient, defined as the partial derivative of the function with respect to  $x$  evaluated at the mean value of measurement  $x$ :

$$c = \left. \frac{\partial f}{\partial x} \right|_{\bar{x}} \quad (10)$$

5. Finally, calculate the expanded uncertainty in  $y$ , symbolized here with a capital  $U_y$ , to give a level of confidence for the interval of uncertainty by multiplying the standard uncertainty in Eq. (9) by the 95% coverage factor, or Student  $t$ -statistic:

$$U_y = \pm t_{95\%, \nu_y} u_y \quad (11)$$

We use the conventional 95% coverage, but are certainly free to use any level. The evaluation of the coverage factor from the  $t$ -distribution requires the degrees of freedom,  $\nu_y$ . We obtain an estimate for the degrees of freedom for the pooled variances in Eq. (9) from the Welch-Satterthwaite formula.<sup>[1]</sup>

$$\nu_y = \frac{u_y^4}{\sum \left[ \frac{(c \cdot u_x)^4}{\nu_x} \right]} \quad (12)$$

The degrees of freedom,  $\nu_x$ , for each variable in Eq. (12) are calculated individually from the Welch-Satterthwaite formula applied to Eq. (5) assuming infinite degrees of freedom for systematic errors, and  $n-1$  degrees of freedom for the random value:

$$\nu_x \cong \frac{u_x^4}{\frac{u_R^4}{n-1} + \frac{u_z^4}{\infty}} \cong \frac{(n-1)u_x^4}{u_R^4} \quad (13)$$

The ubiquitous spreadsheet software Excel is primed to carry out these analysis steps with relative ease using built-in statistical functions and custom macros.<sup>[6]</sup> We illustrate the implementation of the analysis with a simple exercise that students complete in a single class period using inexpensive materials and basic measuring instruments. A second follow-up class period is used to guide students in programming Excel macros designed to simplify and automate the analysis in an Excel worksheet.

## HANDS-ON LEARNING EXERCISE

A simple classroom experiment was devised to allow students to generate data with random and systematic errors for calculating the density of wood from the dimensions of rectangular wooden blocks with a cylindrical hole drilled through their centers, as shown in the schematic of Figure 1. The small wood blocks were rough cut from the same piece of wood, with slight deviations from the mean values of each dimension to introduce noise into the experimental data. This example extends the simpler uncertainty exercise proposed by Yates, who distributed sheets of paper with hand-drawn rectangles to chemistry students for measuring dimensions to calculate

the circumference and area.<sup>[7]</sup>

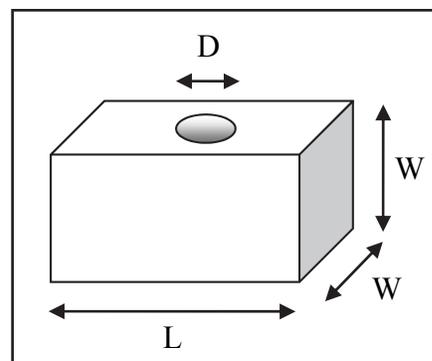
In our wood block exercise, however, students use combinations of analog and digital instruments for measurement. By extending the exercise to a density calculation, we allow for two different types of instruments that must be treated differently in uncertainty analysis.

Students first derive an expression for the density of the block from the ratio of the mass to volume in terms of mass ( $m$ ), length ( $L$ ), width ( $W$ ), and diameter ( $D$ ).

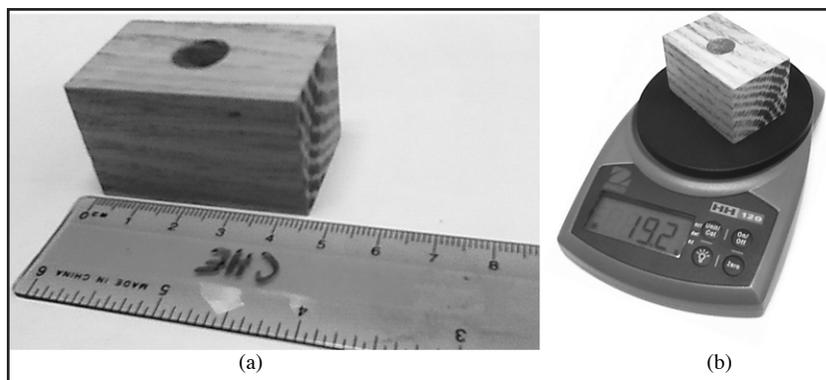
$$\rho = \frac{m}{V} \quad (14)$$

$$V = W \left( LW - \frac{\pi D^2}{4} \right) \quad (15)$$

Students form teams with a minimum size of three members. Each team member is provided with a similar block of wood, an inexpensive plastic ruler for measuring length dimensions in two-unit systems of inches and centimeters, and a portable digital scale for measuring the mass in grams, as shown in Figure 2. The ruler serves as a simple example of an analog instrument. Students must visually interpolate between the graduation marks on the scale for a measurement reading. We have a brief discussion of the issues of reading values from scales using different examples of analog instruments in our labs including liquid thermometers, titration pipettes,



**Figure 1.** Schematic of wooden block with length  $L$ , width  $W$ , and hole diameter  $D$ .



**Figure 2.** (a) Analog ruler for measuring length dimensions. The precision in the graduation marks is 0.1 cm or 1/16 in. The readability is  $\pm 0.05$  cm or  $\pm 1/32$  in (0.03125 in). (b) Digital scale for measuring mass in grams. The precision is 0.1 gm, with a readability of  $\pm 0.05$  gm.

	A	B	C	D	E
1	Team	L/in	W/cm	D/cm	m/gm
2	1	1.8125	2.65	1.12	18.4
3	2	1.8125	2.75	1.15	19.4
4	3	1.78125	2.6	1	18.7
5	4	1.8125	2.85	1.1	18.8
6	5	1.78125	2.7	0.9	18.6
7	6	1.78125	2.7	0.9	19.2
8	7	1.75	2.71	0.96	18.4
9	8	1.8125	2.6	1.1	19.8
10	9	1.8125	2.7	0.95	18.6
11	10	1.8125	2.75	1	18.9
12	11	1.84375	2.7	1.1	18.9
13	12	1.8125	2.75	0.95	18.7

Figure 3. Student-derived experimental data for wooden block dimensions.

TABLE 1 Excel formulas for calculating measurement uncertainty in length. Use similar formulas for the other measurements.				
Variable	Symbol	Cell	Worksheet Formula	Result
Average Length	L	B14	= AVERAGE(B2:B13)	1.8021 in
Number of measurements	n	B15	= COUNT(B2:B13)	12
DoF (n-1) of Average	DoF	B16	= B15-1	11
Standard Deviation	s	B17	= STDEV(B2:B13)	0.0243 in
Random Uncertainty	$u_R$	B18	= B17/SQRT(B15)	0.0070 in
Readability	$\delta$	B19	= 1/32	0.03125 in
Systematic Uncertainty	$u_z$	B20	= B19/SQRT(3)	0.0180 in
Combined Uncertainty	$u_x$	B21	= SQRT(B18 + B20)	0.0194 in
Combined DoF	v	B22	= (B15-1)*(B21/B18)^4	635

Bourdon-tube pressure meters, rotameters for fluid flow rates, manometers, and barometers. The readability of an analog instrument is subject to our comfort level in determining the precision of our interpolation between the graduation marks. The precision of the graduation marks on the ruler is 0.1 cm on one side and 1/16 in on the other, but students report the readability of the ruler as ranging from  $\pm 0.05$  cm down to  $\pm 0.025$  cm or 1/32 in (depending on a student's confidence in his or her ability to interpolate the scale).

The electronic balance is an example of a digital meter

that interpolates the signals for the user. We build on our discussion of how to incorporate uncertainty from examples of digital meters in our labs, including conductivity meters, refractometers, thermocouple digital thermometers, pressure gauges, flow meters, and humidity meters, to name a few. The readability of a digital instrument for use in Eq. (8) is determined from the least significant digit. The precision and readability of the digital balance scale is 0.1 gm, for a uniform systematic uncertainty of  $\pm 0.05/\sqrt{3}$  gm.

All student teams pool their measurement results and record the values in an Excel worksheet, like Figure 3. Although this is a team-based exercise, each individual student creates a copy of the Excel worksheet for personal reference and use later in the program and future lab courses. Students calculate the combined uncertainty in each variable using worksheet functions. To illustrate, we refer to the data for the measurement of length (L) of the block listed in column B on the Excel worksheet, shown in Figure 3. Table 1 contains a summary of the measurement uncertainty calculations including the Excel formulas and worksheet functions for L. By mixing dimension of inches with centimeters we also illustrate how numerical derivative approximations in terms of the input values handle the challenge of unit conversions in uncertainty analysis.

The volume and density of the wood are calculated from the average values for the dimensions and mass in Eqs. (14) and (15). The length units of inches are converted to centimeters

in the volume calculation. Figure 4 shows the results in the Excel worksheet. The uncertainty analysis uses these values for evaluating the derivatives in the sensitivity coefficients.

The analytical formulas for the sensitivity coefficients are listed below:

$$c_L = \frac{\partial \rho}{\partial L} = -m \left( \frac{W}{V} \right)^2 \quad (16)$$

$$c_w = \frac{\partial \rho}{\partial W} = -m \left[ \frac{1}{WV} + \frac{LW}{V^2} \right] \quad (17)$$

$$c_D = \frac{\partial \rho}{\partial D} = \frac{\pi D W m}{2 V^2} \quad (18)$$

$$c_m = \frac{\partial \rho}{\partial m} = \frac{1}{V} \quad (19)$$

The results for  $c$  using the average values of the variables are shown in row 23 of the Excel worksheet in Figure 5.

Unlike the simple linear derivative results of the perimeter and area of a rectangle in the uncertainty exercise of Yates,<sup>[7]</sup> the derivative formulas for density in terms of mass and block dimensions are nonlinear, and prone to formulation errors in a worksheet calculation. At this juncture, it is important to remind the students that uncertainty analysis is by its own nature “uncertain.” High-precision calculations for the sensitivity coefficients are unnecessary. We can take our first step toward simplification of the general process of uncertainty analysis by introducing first-order, finite difference approximations for the sensitivity coefficients:

$$c_i = \frac{\partial \rho}{\partial x_i} \cong \frac{\rho(x_i + \varepsilon) - \rho(x_i)}{\varepsilon} \quad (20)$$

where  $\varepsilon$  is a small value used to perturb the average value of a variable in the formula for density. The perturbation value must be small relative to the magnitude of the variable. In our exercise, students find that a value of  $\varepsilon = 10^{-4}$  is sufficiently small to yield good values for the sensitivity coefficients of each variable. We use the VBA user-defined function in Figure 6 for evaluating the density from the dimensions to simplify the calculations in a worksheet. We use the following worksheet formula to illustrate the calculation of the sensitivity coefficient relative to the wood block length measurement using finite difference derivative approximation:

$$B24 = (\text{DENS}(\$B\$14 + \text{epsilon}, \$C\$14, \$D\$14, \$E\$14) - \text{DENS}(\$B\$14, \$C\$14, \$D\$14, \$E\$14)) / \text{epsilon} \quad (21)$$

```
Public Function DENS(L, W, D, m)
With WorksheetFunction
    DENS = m / (.Convert(L, "in", "cm") * W ^ 2 - W * .Pi() * (D / 2) ^ 2)
End With
End Function
```

Figure 6. VBA user defined function for calculating the density of a wood block.

The epsilon term refers to a named cell on the worksheet that contains a small perturbation value of 0.0001 used in Eq. (20).

By using absolute worksheet cell references for the average values of the variables, we can fill the formula in Eq. (21) across the row of cells in the worksheet for each variable. Then, we simply change the addition of the perturbation value for each variable in turn to get the complete set of sensitiv-

	G	H
1	V/cm <sup>3</sup>	=C14*(CONVERT(B14,"in","cm")*C14-PI()* (D14^2)/4)
2	ρ/(gm/cm3)	=E14/H1

Figure 4. Wood block volume and density formulas evaluated at the average values for the experimental measurements of  $L$ ,  $W$ ,  $D$ , and  $m$ .

ity coefficients. The values in rows 23 and 24 compare the results from the analytical formulas to the finite-difference approximations where we observe good agreement. Finite-difference derivative approximations become important for obtaining good sensitivity coefficients for complicated calculations that involve multiple mathematical operations between the inputs to the output. For working with more complicated problems, we help the students write a VBA

	A	B	C	D	E
1	Team	L/in	W/cm	D/cm	m/gm
14	$x_{ave}$	1.8021	2.71	1.02	18.9
15	$n$	12	12	12	12
16	DoF	11	11	11	11
17	$s$	0.0243	0.0690	0.0900	0.4163
18	$u_R$	0.0070	0.0199	0.0260	0.1202
19	$\delta$	0.03125	0.025	0.025	0.050
20	$u_z$	0.0180	0.0144	0.0144	0.0289
21	$u_x$	0.0194	0.0246	0.0297	0.1236
22	$v$	635.36	25.61	18.84	12.31
23	$c$ analytical	-0.3582	-0.4616	0.0835	0.0320
24	$c$ finite difference	-0.3582	-0.4616	0.0835	0.0320
25	$c$ macro	-0.3582	-0.4616	0.0835	0.0320

Figure 5. Measurement uncertainty and sensitivity coefficients for the wood density exercise, evaluated in an Excel worksheet.

macro like the listing in Figure 7 to automate the evaluation of the sensitivity coefficients.

Students create their macros in the same workbook that contains their worksheet for uncertainty analysis of the wood density calculations. The macro employs input boxes for selecting the ranges of data and writing the results on the worksheet. The algorithm cycles through the variables perturbing the average values one at a time to calculate the derivative approximation. The average value of the variable is reset to its original cell contents before perturbing the next variable for partial derivative evaluations. By default, Excel worksheets automatically recalculate the value of the density function for each small change in the variables. Be sure to turn on automatic calculations if it is not previously set in the Excel worksheet.

With the sensitivity coefficients, we now have all of the information needed to calculate the propagation of the uncertainty in the measurements for the variables (L, W, D, and m) through the calculation of density. First, we calculate the standard uncertainty from the law of propagation of uncertainty. Figure 8 shows the values for the product of the sensitivity coefficient and combined uncertainty required by Eq. (9).

The results for the square of the product ( $c \cdot u$ )<sup>2</sup> guide the stu-

dent in identifying which variable(s) contribute most to the uncertainty in  $y$ . We find that the contribution from the width measurement in column C is larger than the other variables by at least an order of magnitude. We can take steps to reduce the uncertainty in the width by taking more measurements, or using a higher precision ruler. We report the density with standard uncertainty as

$$\rho = 0.603 \pm 0.014 \text{ gm/cm}^3 \text{ (standard uncertainty)} \quad (22)$$

Note that the uncertainty is rounded up. Some practitioners recommend the retention of just one significant figure in the uncertainty, rounded up to  $\rho = 0.61 \pm 0.02 \text{ gm cm}^3$ .

Finally, students calculate the expanded uncertainty for a 95% confidence interval. We need the degrees of freedom from the Welch-Satterthwaite formulas in Eqs. (12) and (13). Table 2 summarizes the worksheet formulas and functions for calculating the propagation of uncertainty and coverage factor. Note how we rounded the result for the combined degrees of freedom down to the nearest integer value for a conservative value of the coverage factor. It is important to help students learn to match the precision in their reported

Variable	Symbol	Cell	Worksheet Formula	Result
Standard Uncertainty	$u_y$	B28	=SQRT(SUMSQ(B26:E26))	0.014 gm/cm <sup>3</sup>
Degrees of Freedom	$\nu_y$	B30	=ROUNDDOWN((B28^4)/(SUM(B29:E29)),0)	58
Coverage Factor	$t$	B31	=TINV(0.05,B30)	2.002
Expanded Uncertainty	$U_y$	B32	=ROUNDUP(B28*B31,3)	0.029 gm/cm <sup>3</sup>

```
Public Sub dfdx()
' Get sensitivity coefficients from finite difference derivative approximations

' Get input from the worksheet
With Application
    Set x = .InputBox(Type:=8, prompt:="Select range of average variables:")
    Set f = .InputBox(Type:=8, prompt:="Select cell with function result:")
    Set c = .InputBox(Type:=8, prompt:="Select range for sensitivity coefficients:")
End With

' Specify number of variables, perturbation factor & save function value
n = x.Count: eps = 0.0001: finit = f

' Loop through variables to calculate sensitivity coefficients
For j = 1 To n
    temp = x(j).Formula      ' save worksheet formula for average variable j
    x(j) = x(j) + eps        ' perturb value of variable j in the worksheet
    c(j) = (f - finit) / eps ' calculate the sensitivity coefficient for variable j
    x(j) = temp              ' replace value/formula of variable j in the worksheet
Next j
End Sub
```

**Figure 7.** VBA macro for calculating sensitivity coefficients in an Excel worksheet.

result to the precision in the uncertainty. We now have the expanded uncertainty for wood density:

$$\rho = 0.61 \pm 0.03 \text{ gm/cm}^3 \text{ (95\% confidence)} \quad (23)$$

or

$$\rho = 0.61 \text{ gm/cm}^3 \pm 5\% \text{ (95\% confidence)} \quad (24)$$

Figure 9 shows the Excel worksheet results for expanded uncertainty in density.

### JITTER MACRO

A macro named JITTER that automates the complete analysis for uncertainty propagation is provided to the students.<sup>[4]</sup> The macro incorporates the VBA code from the student-generated macros from the class exercises. To use the macro, students must set up a worksheet with a cell containing the final value of the function ultimately calculated from the average values of experimental inputs. In addition, the worksheet must include ranges of values for the average variables, random uncertainties, readability, and degrees of freedom. In most cases, the degrees of freedom for averaged values are the number of replicated experiments, less one (n-1). An example of a different number of degrees of freedom is the use of least-squares regression parameters, such as the slope or intercept of a line, where the degrees of freedom are the number of regression data points less two. The macro uses input boxes to prompt the user for the required information. The input boxes displayed in Figure 10 show the cell and range addresses used for the wood density exercise with the density calculated in Cell B34 on the worksheet.

The JITTER macro output for our class exercise

in Figure 11 shows the absolute and relative expanded, standard, and maximum standard uncertainties. In addition, the output includes the sensitivity coefficients and contributions from each variable to the propagation of uncertainty. We see that variable 2, corresponding to width, contributes approximately 65% of the uncertainty and should be a target for refining the experiments to reduce uncertainty in our density

	A	B	C	D	E
1	Team	L/in	W/cm	D/cm	m/gm
26	$c \cdot u_x$	-0.007	-0.011	0.002	0.004
27	$(c \cdot u_x)^2$	4.81E-05	1.29E-04	6.16E-06	1.56E-05
28	$u_y$	0.014			

Figure 8. Excel worksheet calculation of standard uncertainty.

	A	B	C	D	E
1	Team	L/in	W/cm	D/cm	m/gm
29	$(c \cdot u_x)^4 / v$	3.64E-12	6.48E-10	2.01E-12	1.98E-11
30	$v_y$	58			
31	t	2.002			
32	$U_y$	0.029			

Figure 9. Excel worksheet with results for expanded uncertainty in density.

Figure 11. Results from macro JITTER for the propagation of uncertainty in the wood density exercise.

$U_{95\%} = \pm$	2.8E-02	$= \pm$	4.7%
$u_{\max} = \pm$	2.5E-02	$= \pm$	4.1%
$u = \pm$	1.4E-02	$= \pm$	2.3%
$t_{95\%} =$	2.00	DoF =	58
$c_1 =$	-3.58E-01	$(c \cdot u_1)^2 =$	24.2%
$c_2 =$	-4.62E-01	$(c \cdot u_2)^2 =$	64.8%
$c_3 =$	8.35E-02	$(c \cdot u_3)^2 =$	3.1%
$c_4 =$	3.20E-02	$(c \cdot u_4)^2 =$	7.9%

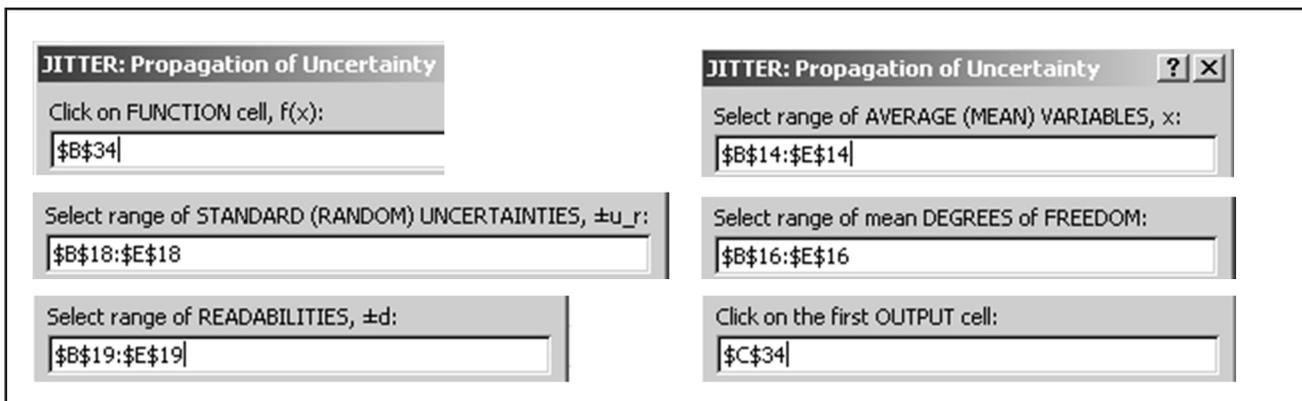


Figure 10. JITTER macro input boxes with worksheet ranges for wood density exercise.

calculation. An Excel workbook with the data and calculations for the example exercise, as well as the general-purpose JITTER macro, is available for download from the author's website (<[www.d.umn.edu/~rdavis/cee](http://www.d.umn.edu/~rdavis/cee)>).

## RESULTS

Students develop tools in these exercises that are applicable to a wide variety of problems involving uncertainty analysis in their remaining coursework, but particularly in our program's unit operations lab courses. Armed with an understanding of the basic assumptions and data requirements, students may use other software tools for uncertainty analysis available freely on the Internet or commercially.

We evaluate our students' ability to transfer their skill with the formulas and Excel tools for uncertainty analysis by an exam question where they are allowed to use their Excel worksheets and macros from the class exercises. We find that over 90% of the students are consistently able to transfer their uncertainty analysis skills from the class exercise to correctly analyze the uncertainty in the exam problem.

We also see improvements in uncertainty analysis conducted by students in our lab courses. Evidence includes an increased discussion and application of analysis of uncertainty in lab reports with more appreciation for reporting reliability of experimental results. Students are also careful to include estimates of uncertainty in all final reported values. Although we warn the students that they will need these tools in courses later in the program, a few students have a habit of compartmentalizing their learning. Once they are finished with the computational methods course, they do not automatically make the connection of applying the analysis later in the program. However, when we provide them with a copy of the wood density exercise, they are able to quickly relearn and apply these skills.

In one particularly complicated lab analysis, students use the macro to estimate the uncertainty in the calculation of chemical equilibrium constants. Experimental measurements of volume and mass to calculate reagent concentration and extent of reaction are fed into a worksheet set up to perform a complex series of stoichiometric mass conservation calculations to arrive at a value of the equilibrium constant. These calculations may be spread across multiple worksheets. Propagating the uncertainty through these involved series of calculations by hand is tedious and prone to mistakes. The JITTER macro efficiently performs the analysis requiring little effort setting up the worksheets for the macro inputs. The students also appreciate the additional VBA programming skills developed as part of this exercise and report using this additional skill in academic and industrial settings.

## CONCLUSIONS

A simple hands-on active-learning exercise provides students with experience collecting and analyzing experimental data with random and fixed uncertainties. The exercise uses

data collected from analog and digital instruments that give the students experience interpreting fixed uncertainties from readability. Students create an Excel worksheet that serves as a template of the steps for calculating the propagation of measurement uncertainty in engineering analysis. This worksheet template, coupled with VBA macros, gives the students powerful tools for including measures of reliability in their calculated results. By making the process simple, relatively easy, and painless, students who formerly avoided uncertainty analysis are now more willing and able to conduct proper uncertainty analysis. Similar tools may be developed in other computational platforms, such as Matlab or Python, as needed by a department to meet the needs of its constituents. We elected to focus on Excel based on the feedback we receive from our own graduates who report using Excel with macros as their primary computational software.

## NOMENCLATURE

c	= sensitivity coefficient
D	= diameter, cm
f	= relationship between input variables and output variable
L	= length, cm
m	= mass, gm
n	= number of replicated measurements
s	= sample standard deviation
t	= coverage factor, or Student's t-statistic
$U_y$	= expanded uncertainty in the output value y
$u_R$	= random uncertainty or standard error
$u_x$	= combined uncertainty in measurements
$u_y$	= standard uncertainty in the output value y
$u_z$	= systematic uncertainty degrees of freedom
$\bar{V}$	= volume, cm <sup>3</sup>
$v_x$	= degrees of freedom
$v_y$	= degrees of freedom for pooled variances in expanded uncertainty
W	= width, cm
x	= input variable from experimental measurements
y	= output value calculated from input values
$\delta$	= readability
$\epsilon$	= perturbation in x
$\rho$	= density, gm/cm <sup>3</sup>

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