

SPREADSHEET SOLUTIONS TO TWO-DIMENSIONAL HEAT TRANSFER PROBLEMS

RONALD S. BESSER

Louisiana Tech University • Ruston, LA 71272

Students in the undergraduate heat transfer class seem to become more excited about the subject when they begin solving realistic problems that somehow connect to their experience. While many of these problems can be solved using the approximations of one-dimensional symmetry, a large body of interesting and relevant problems must be tackled with two-dimensional (2-D) methods. This paper describes a simple method for solving these problems using any of a number of spreadsheet programs, such as Microsoft Excel, Corel Quattro Pro, Lotus 1-2-3, etc. We have successfully used this method in junior-level heat transfer at Louisiana Tech University for the past two years.

TWO-DIMENSIONAL METHODS

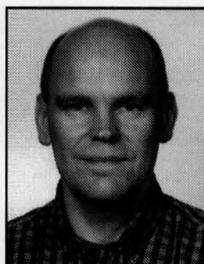
Various approaches are available for solving 2-D problems. Analytical solutions to engineering problems are highly desirable due to the elegant connection that becomes visible between physical and mathematical principles. For a few simple geometries, methods such as separation of variables^[1] can be applied, or solutions to characteristic differential equations may be available,^[2] but they cover only a small fraction of the possible problems.

Graphical methods^[3,4] have been used for many years to produce solutions for situations requiring qualitative or approximate answers. Information about these methods is available in several textbooks, but graphical techniques may be perceived as excessively approximate compared to the numerical methods that are so accessible today. Although no statistics to this effect are known, the sense is that the graphical methods are seldom taught.

The explosion of the information age has provided ready access for engineers and students to high-powered desktop

The method can . . . be extended by solving problems with time dependence (transient problems), problems with geometries that would benefit from rectangular rather than square elements, geometries with edges at oblique angles, and even three-dimensional problems.

machines that are suited for numerical solutions to heat transport and other engineering problems. While commercial software such as ANSYS, PDEase, FlexPDE, etc., can tackle two- and even three-dimensional problems, extremely useful 2-D solutions using the Finite Difference Method (FDM) can be easily obtained by students or engineers with an ordinary spreadsheet. Furthermore, the process of setting up the problem, including formulating the boundary conditions, laying out the geometry statement, determining the convergence conditions, etc., reinforces the understanding of heat transfer principles by the student. Obtaining the solution by this process also promotes understanding of how commercial solvers work.



Ronald S. Besser has been Associate Professor of Chemical Engineering at the Louisiana Tech University Institute for Micro-manufacturing since 1999. He holds a BS in chemical engineering from U.C. Berkeley, and an MS and PhD in materials science and engineering from Stanford University. His research and development interests are in chemical-MEMS and sensors, thin-film materials, plasma deposition and etching, sub-micron processing, device physics, and characterization.

The value of convenient spreadsheet programs for solving a variety of chemical engineering problems has been previously demonstrated in various sources.^[5] While the description here applies solely to heat transfer, a nearly identical approach can be used to solve problems of mass transfer, fluid flow, electric current flow, mechanical stress, etc., because of analogous mathematical descriptions.

BASIS OF FINITE DIFFERENCE METHOD

The FDM starts by taking the system under study and dividing it into a large (but “finite!”) number of rectangular elements. Each element is assumed to be isothermal, *i.e.*, the entire element exists at a single temperature. At the center of each element is a “node” or “mesh point” with a unique identifier based on its position in the “nodal network” or “mesh.” Integer subscripts (m,n) relate to position on an x-y axis system with a discrete value range. Figure 1 displays this setup.

In order to solve for the temperatures in the system, we need temperature derivatives for insertion into the heat equation. Consider the temperature $T_{m,n}$ of an arbitrary element (m,n) as shown in Figure 1. The first derivatives (in x and y) are written by assuming linear variation of temperature between node points. Since second derivatives are just first derivatives of first derivatives,

$$\frac{\partial^2 T}{\partial x^2} \Big|_{m,n} \approx \frac{\frac{\partial T}{\partial x} \Big|_{m+\frac{1}{2},n} - \frac{\partial T}{\partial x} \Big|_{m-\frac{1}{2},n}}{\Delta x} \approx \frac{\frac{T_{m+1,n} - T_{m,n}}{\Delta x} - \frac{T_{m,n} - T_{m-1,n}}{\Delta x}}{\Delta x} = \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2} \quad (1)$$

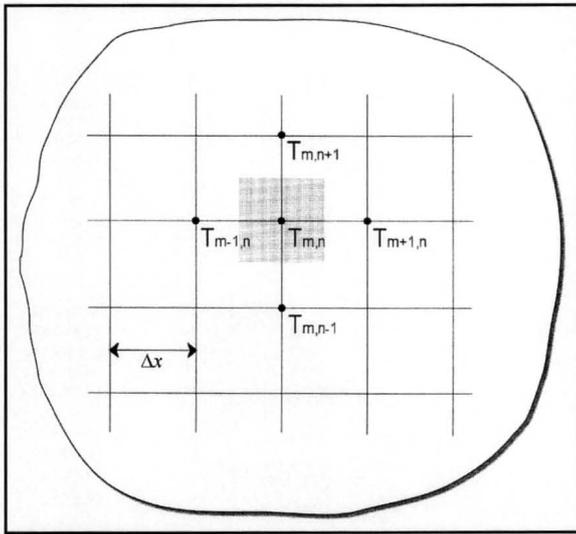


Figure 1. Mesh representing the body or system under study. $T_{m,n}$ is the uniform temperature of the shaded element.

Similarly, for the vertical direction

$$\frac{\partial^2 T}{\partial y^2} \Big|_{m,n} \approx \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2} \quad (2)$$

We know the three-dimensional heat equation that relates conductive fluxes and heat generation to the time rate of change of the temperature of a system as

$$\frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) + \dot{q} = \rho C_p \frac{\partial T}{\partial t} \quad (3)$$

We can apply some simplifying assumptions that nevertheless hardly reduce the usefulness of the equation by imposing steady-state conditions, no generation, and a thermal conductivity that is temperature independent. The result is LaPlace’s equation, which, when we make the additional assumption of 2-D symmetry, *i.e.*, T is constant in z, is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (4)$$

Inserting the second derivatives (Eqs. 1 and 2) into Eq. (4), taking the case of a square mesh (*i.e.*, $\Delta x = \Delta y$, assumed throughout the rest of this article), and doing some algebra to solve for $T_{m,n}$, we get

$$T_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1}}{4} \quad (5)$$

In other words, the temperature of interest of a location within the “bulk” of the system, and not at a boundary, is just the average of the four temperatures surrounding it.

SIMPLE EXAMPLE: CONSTANT BOUNDARY TEMPERATURE

We can apply this result immediately to solve a real problem. Consider a metal plate 0.9 m x 0.9 m in size that has its edges held at constant temperatures, as shown in Figure 2. We ask, what is the temperature field that develops in the

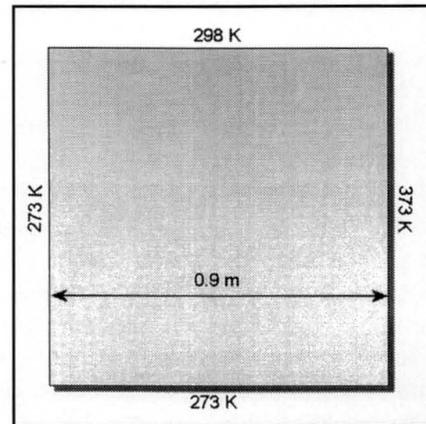


Figure 2. Metal plate 0.9 m x 0.9 m in size with edges held at constant temperatures.

plate once steady-state conditions are attained? We set up a simple spreadsheet, as shown in Figure 3, with a cell representing the nodal temperature of each 0.1 m x 0.1 m element in the plate. The nodes are at the center of each element, except at the boundaries. The boundary nodes sit at the edge of their elements, and the elements are half the size of the nodes in the center portion (note that the corner elements are one-fourth the size of the central elements).

The spreadsheet is set up by first turning off any limitations on circular references. In Excel, this is done by going to Options and selecting the Calculation tab, then choosing manual calculation. The number of iterations and convergence criteria are also set there. *These are important steps*, as without them, Excel will return errors when copying the nodal equations, leading to untold frustration.

Now the perimeter temperatures can be input as constants. Then the nodal equations are entered at the interior points. The equation for cell B9, for example, is

$$=(A9+B8+C9+B10)/4 \quad (6)$$

Once typed into B8, the equation can then simply be copied and pasted to the rest of the interior cells.

After setting up the equations, hitting F9 causes the spreadsheet to calculate a number of times set by the calculation limit or the convergence limit ("maximum change" as labeled in the Excel Calculation option) entered previously. Repeated presses of F9 guarantee that the solution converges before reaching the number of iterations limit.

This problem converges almost immediately (in 80 iterations) using a maximum change criterion of 0.001. Reducing the size of this convergence limit will result in a higher precision solution at the cost of increased CPU time. The accuracy of the solution depends on how closely the mesh approximates the actual geometry. In general, accuracy improves as the node spacing decreases. Accuracy can be checked by halving the node spacing and recalculating a solution. The calculation has reached its highest accuracy if the two solutions are found to be essentially the same. If substantial difference exists, the process of reducing node spacing and recalculating is continued until the difference diminishes.

The solution, shown in Figure 4, was also graphed using the surface plot option in Excel. The plot gives an excellent view of what is going on with the plate's temperature field. Higher spatial resolution could be obtained by setting a smaller increment size, resulting in a larger number of cells. Though execution time trades off with resolution, even highly resolved arrays iterate quickly with a current-model PC. Moreover, the linear nature of the equations being evaluated tends to prevent the occurrence of computational instabilities that sometimes appear with iterative methods.

| | A | B | C | D | E | F | G | H | I | J | K |
|----|--|-----|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1 | Metal Plate with Edges at Constant Temperature | | | | | | | | | | |
| 2 | | | | | | | | | | | |
| 3 | k | 10 | W/m/K | | | | | | | | |
| 4 | delx | 0.1 | m | | | | | | | | |
| 5 | | | | | | | | | | | |
| 6 | | | | | | | | | | | |
| 7 | | | | | | | | | | | |
| 8 | 298 | 298 | 298 | 298 | 298 | 298 | 298 | 298 | 298 | 298 | |
| 9 | 273 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 373 | |
| 10 | 273 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 373 |
| 11 | 273 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 373 |
| 12 | 273 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 373 |
| 13 | 273 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 373 |
| 14 | 273 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 373 |
| 15 | 273 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 373 |
| 16 | 273 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 373 |
| 17 | 273 | 273 | 273 | 273 | 273 | 273 | 273 | 273 | 273 | 273 | |
| 18 | | | | | | | | | | | |

Figure 3. Spreadsheet set up to solve for the temperature distribution of the plate in Figure 2. The interior nodes consist of formulae (as shown for B9), while the perimeter values are constants, as shown.

DERIVING EQUATIONS FOR BOUNDARY CONDITIONS

The above example is especially easy because of the constant boundary conditions (BCs). Changes in heat transfer mode (e.g., convection or radiation from a solid) or a change in material (transition to a region of different thermal conductivity) necessitate more complicated equations in the boundary cells. Several textbooks list these boundary conditions for various cases.^[6,7] The ability to derive arbitrary BC equations,^[8,9] however, gives one the confidence to attack a variety of problems.

The basic approach to deriving a BC equation is to perform a heat balance on the boundary element of interest. Since we have assumed the absence of generation, this amounts to

$$\sum q_{in} = 0 \quad (7)$$

where the q_{in} are rates of heat transfer from adjoining cells.

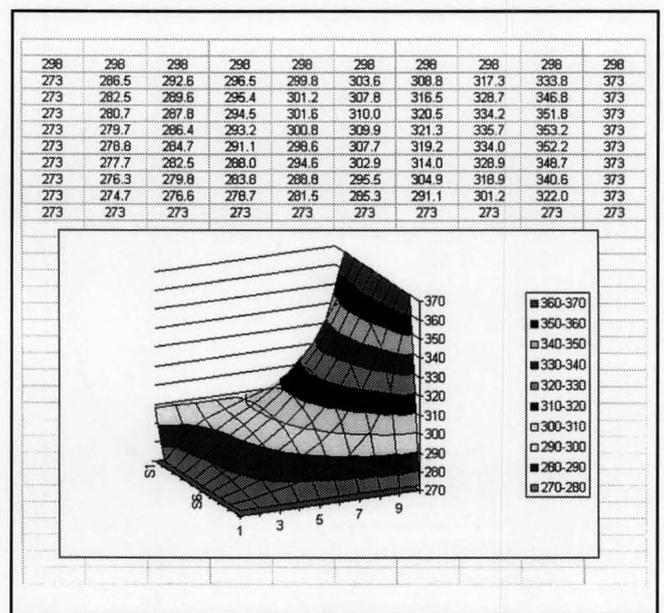


Figure 4. Solution of slab problem after 80 iterations.

A surprising array of complicated problems can be solved using this method that cannot be directly solved with analytical methods.

This can be illustrated by example. Consider convection above a horizontal surface, as illustrated in Figure 5. Based on the figure and Eq. (7),

$$q_1 + q_2 + q_3 + q_c = 0 \quad (8)$$

Some students may see a comfortable analogy between this equation and Kirchoff's current law in electric circuits, *i.e.*, the currents entering a circuit node must sum to zero. Now, using Fourier's Law for the conduction rates,

$$q_1 = -kA \frac{dT}{dx} \approx -kA \frac{\Delta T}{\Delta x} \quad (9)$$

and the standard expression for the convection transport rate

$$q_c = hA\Delta T \quad (10)$$

we have

$$k \left(\frac{\Delta x}{2} \right) (1m) \frac{(T_{m-1,n} - T_{m,n})}{\Delta x} + k(\Delta x)(1m) \frac{(T_{m,n-1} - T_{m,n})}{\Delta x} + k \left(\frac{\Delta x}{2} \right) (1m) \frac{(T_{m+1,n} - T_{m,n})}{\Delta x} + h\Delta x(1m)(T_\infty - T_{m,n}) = 0 \quad (11)$$

For this equation, we have assumed an arbitrary depth of the system of 1 m. In the 2-D symmetry that we have adopted, all properties of the system, including its structure and temperature, are constant with depth (*i.e.*, the z direction). The arbitrary choice of 1 m simplifies the arithmetic and permits calculated secondary quantities to be considered on a perimeter-of-depth basis. The flux "faces" at the left and right ends of the cell are only a half-width tall since the element sits at an edge. The Fourier Law terms are cast to have a positive sign by listing the T terms in the order of exterior temperature minus interior temperature with respect to the element being analyzed.

After canceling and solving for the node temperature, we get

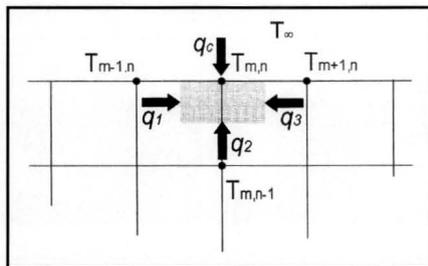


Figure 5. Schematic for deriving boundary equation for the case of convection above a horizontal surface.

$$T_{m,n} = \frac{1}{2 \left(2 + \frac{h}{k} \Delta x \right)} \left(T_{m+1,n} + 2T_{m,n-1} + T_{m-1,n} + \frac{2h}{k} \Delta x T_\infty \right) \quad (12)$$

This equation is inserted into the corresponding spreadsheet cell. References to the convective heat transfer coefficient (h), thermal conductivity (k), element width (Δx), and fluid temperature (T_∞) can be made by giving these variables names, making it easier to transcribe and debug equations.

GENERATION EFFECTS

The effects of heat generation may be included by adding a term to the heat balance of an element. This applies to balances done on both edge cells and interior cells. Consider first the edge cell with convection from its top surface that we analyzed above. With the generation term, Eq. (7) becomes

$$\sum q_{in} + \dot{q}V = 0 \quad (13)$$

where \dot{q} is the volumetric rate of heat generation (W/m^3) that is considered to be uniform within the element. The volume of the element is given by V. Now Eq. (13) becomes

$$q_1 + q_2 + q_3 + q_c + \dot{q}(\Delta x) \left(\frac{\Delta x}{2} \right) (1m) = 0 \quad (14)$$

After making the substitutions we made above and applying some algebra, the nodal temperature is found to be

$$T_{m,n} = \frac{1}{2 \left(2 + \frac{h}{k} \Delta x \right)} \left[T_{m+1,n} + 2T_{m,n-1} + T_{m-1,n} + \frac{2h}{k} \Delta x T_\infty + \dot{q} \frac{(\Delta x)^2}{k} \right] \quad (15)$$

Similarly, applying this approach to an interior element yields an equation analogous to Eq. (5)

$$T_{m,n} = \frac{T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} + \dot{q} \frac{(\Delta x)^2}{k}}{4} \quad (16)$$

By assigning different generation rates to different areas within the system, problems involving *non-uniform* generation can be attacked. Problems of this type are in general very difficult to solve. Because of its relative simplicity, this method can help one set up and solve many situations involving non-uniform generation and thereby build intuition about this class of problem.

CALCULATING THE HEAT TRANSFER RATE FROM THE TEMPERATURE FIELD

Fourier's law written for 2-D permits the determination of the heat rate at an arbitrary point given knowledge of the spatial temperature distribution. The heat rate can be treated as a vector quantity and solved at an arbitrary direction by this means, but for the purposes of junior-level heat transfer, it is sufficient to determine fluxes that are parallel to the x- and y-axes.

Figure 6 shows an interior element where conduction is occurring. We know the heat transfer rate from element A to element B from Fourier's law by

$$q_{A \rightarrow B} \approx -kA \frac{\Delta T}{\Delta x} = k\Delta x(1m) \frac{T_A - T_B}{\Delta x} \quad (17)$$

By now applying this formula along a vertical or horizontal length of several elements, and adding up the heat rate contributions of all these elements, we can calculate the total rate across any plane in the system. Not only is this capability useful for determining heat transfer rates for specific cases, but it also permits one to check the self-consistency of any solution by making sure that the heat balance around any boundary is correct.

The perimeter heat fluxes of the metal slab above were calculated as shown in the spreadsheet of Figure 7. Here the heat rates from each side of the body have been determined by first calculating the rates on a cell-by-cell basis and then adding them up. As we see from the figure, the heat rates from all the sides, when summed, add to zero—as they should for a body in steady-state without generation.

EXAMPLE: HEATING DUCTS IN THE FLOOR

We further illustrate the method by solving a more practical, real-life problem taken from our heat transfer course. The problem shows the utility of this method, as solutions by other routes would only come with much difficulty. The problem is stated this way:

A large, horizontal fiberglass slab serving as a floor is heated by hot air passing through ducts buried in it, as shown in the cross-section in Figure 8, where $S = 160$ mm. The square ducts are centered in the fiberglass, which is exposed to the ambient above and insulating earth below. For the case with the top surface and duct surfaces at 25 and 85°C, respectively, calculate the heat rate from each duct, per unit length of duct. The thermal conductivity of the fiberglass is 2.5 W/(mK).

The spreadsheet solution to this problem is shown in Figure 9, with key equations spelled out for clarity. An area in

the upper left allows input parameters (whose cells are named as variables) to be changed so that the problem can be solved for a variety of cases. The solution exploits the inherent symmetry of the problem by solving only a segment of the system, yielding the heat rate for a half-duct, which is then doubled to arrive at an answer.

A detail of the spreadsheet shown is the labeling of the

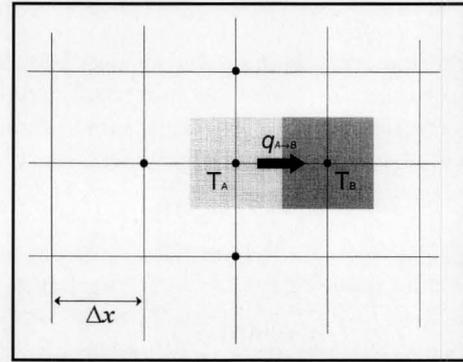


Figure 6. Heat transfer rate from an internal element A into another internal element B.

| | | | | | | | | | |
|------------------|----------|----------|----------|----------|----------|----------|----------|-----------|---------------|
| 290 | 290 | 290 | 290 | 290 | 290 | 290 | 290 | 290 | 290 |
| 273 | 266.5 | 262.8 | 256.5 | 259.8 | 303.8 | 308.8 | 317.3 | 333.8 | 373 |
| 273 | 262.5 | 269.8 | 295.4 | 301.2 | 307.8 | 308.8 | 316.5 | 328.7 | 346.8 |
| 273 | 260.7 | 267.8 | 294.5 | 301.5 | 310.0 | 320.5 | 334.2 | 351.8 | 373 |
| 273 | 259.7 | 268.4 | 293.5 | 300.8 | 309.9 | 321.3 | 336.7 | 353.2 | 373 |
| 273 | 278.8 | 284.7 | 291.1 | 298.6 | 307.7 | 319.2 | 334.0 | 352.2 | 373 |
| 273 | 277.7 | 282.5 | 289.0 | 294.8 | 302.9 | 314.0 | 328.9 | 348.7 | 373 |
| 273 | 276.3 | 279.8 | 283.8 | 288.8 | 296.5 | 304.9 | 318.9 | 340.6 | 373 |
| 273 | 274.7 | 276.6 | 278.7 | 281.5 | 285.3 | 291.1 | 301.2 | 322.0 | 373 |
| 273 | 273 | 273 | 273 | 273 | 273 | 273 | 273 | 273 | 273 |
| Q _{top} | 114.7181 | 53.52518 | 15.256 | -19.1425 | -56.1978 | -108.489 | -189.31 | -357.8634 | total 599.474 |
| Q _{bot} | -17.1386 | -36.6022 | -57.1459 | -84.5192 | -122.535 | -180.905 | -267.438 | -489.718 | -1270 |
| Q _{lhs} | | | | | | | | | |
| | -136.262 | | 392.137 | | | | | | |
| | -84.653 | | 261.859 | | | | | | |
| | -77.457 | | 212.157 | | | | | | |
| | -67.167 | | 198.346 | | | | | | |
| | -67.653 | | 207.844 | | | | | | |
| | -45.518 | | 243.106 | | | | | | |
| | -32.544 | | 323.865 | | | | | | |
| | -17.137 | | 610.252 | | | | | | |
| total | 528.82 | | 2349.297 | | | | | | |

$\sum q_{in} = q_{top} + q_{bot} + q_{lhs} + q_{rhs}$
 $Q_{top} + Q_{bot} + Q_{lhs} + Q_{rhs} = 0.000$

Figure 7. Determination of heat rates from all sides of the metal slab pictured in Figure 2. The sum of the rates equals zero, verifying the solution of the temperature distribution.

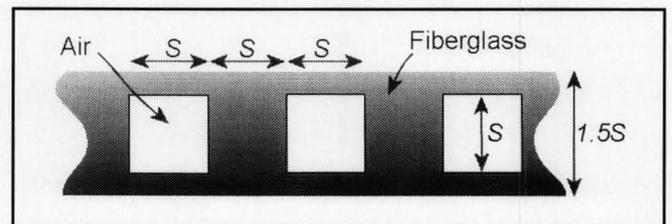


Figure 8. Fiberglass slab with embedded ducts for heating by air flow. The top surface of the slab is a room floor, while the bottom surface lies against insulating earth.

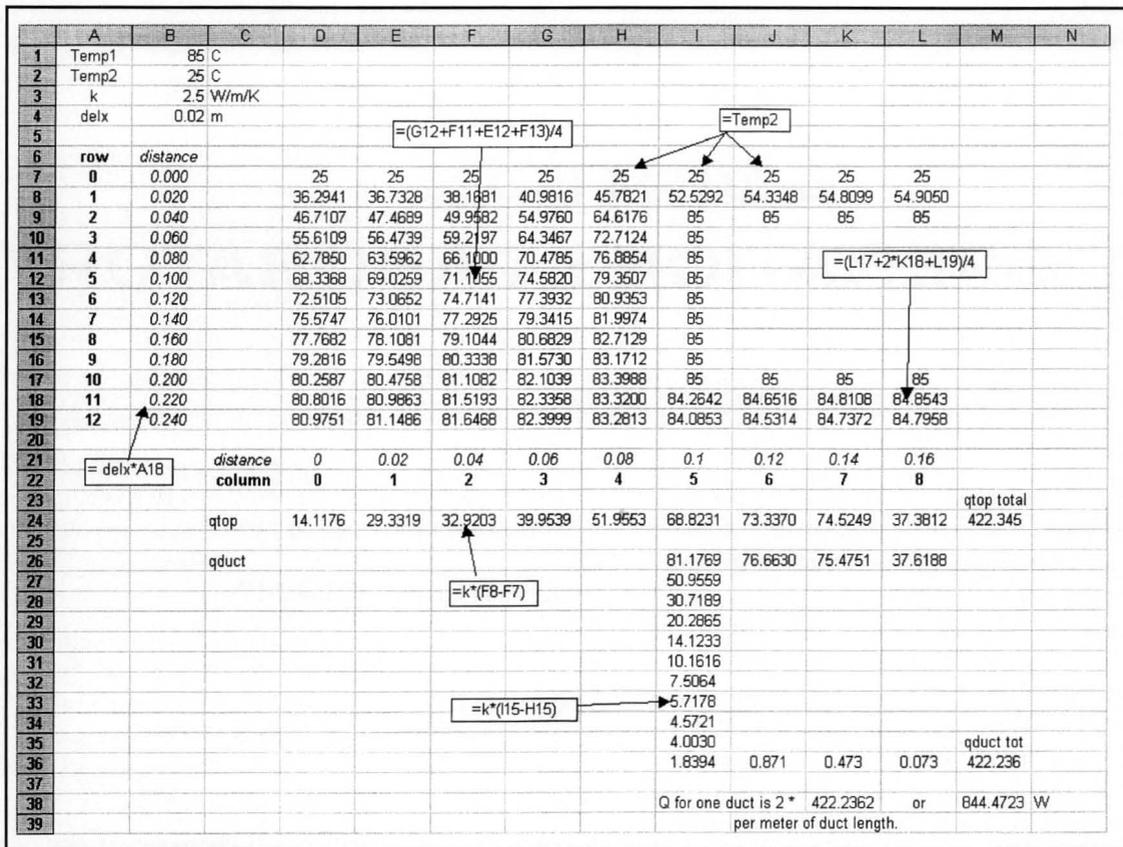


Figure 9.

Solution to fiberglass slab problem. The heat rate from each duct is 844 W.

nodal rows and columns and their locations in vertical and horizontal directions. These labels help one to keep track of the geometry of the problem.

The lower portion of the sheet (row 24 down) shows the calculation of heat rates at nodes along the floor or top surface (“qtop”) and along the duct perimeter (“qduct”). The close agreement of these heat rates serves as a check on the answer and reinforces the notion that input and output rates must be equal in a steady-state problem.

CONCLUSION

We have described here a computer-aided, finite-difference approach to solving 2-D heat transfer problems in the undergraduate curriculum. A surprising array of complicated problems can be solved using this method that cannot be directly solved with analytical methods. Once learned, the method is applicable to a number of other course situations. For example, our students in unit operations laboratory have solved for heat transfer characteristics of critical but odd-shaped components in heat transfer equipment under study. The method can also be extended by solving problems with time dependence (transient problems), problems with geometries that would benefit from rectangular rather than square elements, geometries with edges at oblique angles, and even three-dimensional problems.

ACKNOWLEDGMENTS

The author thanks the students of Chemical Engineering 313 for their interaction with and feedback on the methods described here. He also thanks Dr. James Palmer and Mr. Rohit Ghan for reviewing the manuscript.

REFERENCES

1. Boyce, W.E., and R.C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 2nd ed. John Wiley and Sons, New York, NY; p. 423 (1969)
2. Jaeger, J.C., and H.S. Carslaw, *Conduction of Heat in Solids*, 2nd ed., Oxford University Press, Oxford, England (1986)
3. Welty, J.R., C.E. Wicks, R.E. Wilson, and G.L. Rorrer, *Fundamentals of Momentum, Heat, and Mass Transfer*, 4th ed., John Wiley and Sons, New York, NY; p. 246 (2001)
4. Incropera, F.P., and D.P. DeWitt, *Fundamentals of Heat and Mass Transfer*, 4th ed., John Wiley and Sons, New York, NY; p. 167 (1996)
5. See, for example, B.S. Gottfried, *Spreadsheet Tools for Engineers*, Excel 97 Version, WCB/McGraw-Hill, Boston (1998); also see various articles in *Chemical Engineering Education*, e.g., M. Misovich and K. Biasca, “The Power of Spreadsheets in a Mass and Energy Balances Course,” *Chem. Eng. Ed.*, **25**(1), 46 (1991) and P.E. Savage, “Spreadsheets for Thermodynamics Instruction,” *Chem. Eng. Ed.*, **29**(4), 262 (1995)
6. Reference 4, page 178
7. Mills, A.F., *Heat and Mass Transfer*, Irwin, Chicago, IL, p. 131, 197 (1995)
8. Chapra, S.C., and R.P. Canale, *Numerical Methods for Engineers*, 3rd ed., McGraw-Hill Book Company, Boston, MA; p. 820 (1998)
9. Arpacı, V.S., S.-H. Kao, and A. Selamet, *Introduction to Heat Transfer*, Prentice-Hall, Upper Saddle River, NJ; p. 194 (1999) □