NEWTON'S LAWS, EULER'S LAWS, AND THE SPEED OF LIGHT

STEPHEN WHITAKER
University of California at Davis • Davis, CA 95616

Truesdell[1] tells us that Newton listed his three laws of motion as:

Newton (1642-1727)

I. *Every body continues in its state of rest, or of uniform motion straight ahead, unless it be compelled to change that state by forces impressed upon it.*

II. *The change of motion is proportional to the motive force impressed, and it takes place along the right line in which the force is impressed.*

III. *To an action there is always a contrary and equal reaction; or, the mutual actions of two bodies upon each other are always directed to contrary parts.*

Truesdell[2] also tells us that Newton never presented these ideas in the form of equations, and because of this there are differences to be found in the literature. In this work we choose "motion" to mean mass times velocity, $m\underline{v}$, and we choose "motive force" to be represented by $\underline{f}$. This leads to

$$\text{Newton I:} \qquad m\underline{v} = \text{constant}, \quad \underline{f} = 0 \tag{1}$$

while the second law takes the form

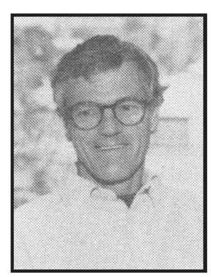
$$\text{Newton II:} \qquad \frac{d}{dt}\left(m\underline{v}\right) = \underline{f} \tag{2}$$

Here the "change of motion" has been interpreted as the time rate of change of the momentum, $m\underline{v}$. Often a precise definition of $\underline{v}$ is not given in the discussion of Newton's first and second laws, and we will return to this matter in subsequent paragraphs. Clearly Newton's first law is a special case of Newton's second law, and one can wonder why it was stated as an independent law. Physicists[3-5] have pointed out that Eq. (1) was deduced earlier by Galileo (1564-1642), thus Newton was motivated to elevate this result to the position of a "law."

Newton's third law for two interacting bodies can be expressed as

$$\text{Newton III:} \qquad \underline{f}_{12} = -\underline{f}_{21} \tag{3}$$

in which $\underline{f}_{12}$ is the force that body #2 exerts on body #1, and $\underline{f}_{21}$ is the force that body #1 exerts on body #2. The most dramatic success of these laws was their use, along with the *law of gravitational attraction*, to justify Kepler's three



Stephen Whitaker *is a professor emeritus at the University of California, Davis. His interests in engineering education are directed toward avoiding leaps of faith by building upon material studied in prerequisite courses. He has received various departmental teaching awards in addition to the Tau Beta Pi Outstanding Teacher Award (College of Engineering), the Magnar Ronning Award for Teaching Excellence (UC Davis), the Distinguished Teaching Award (UC Davis), the Engineering Alumni Distinguished Teaching Award (College of Engineering), and the Warren K. Lewis Award (AIChE).*

empirical laws of planetary motion. In a careful statement of Newton's laws, one often notes that they are valid in an *inertial frame*. This naturally leads to the question: What is an inertial frame? The answer is that an inertial frame is a frame in which Newton's laws are valid! We can only escape from this circular argument by noting that an inertial frame must be determined by experiment.[6] In Newton's case, the verification of Kepler's laws indicated that the sun and the "fixed stars" represented a good approximation of an inertial frame for the study of planetary motion.

If we think about applying Eq. (2) to the motion of a body, we must wonder what is meant by the velocity, $\underline{v}$, since all parts of a body need not have the same velocity. Physicists often deal with this problem by arguing that Eq. (2) applies to "mass points" that are small enough so that their motion can be described by a single velocity. The statement that something is "small" always leads to the question: Small relative to what? Feynman, *et al.*,[7] touch on this problem by considering the cloud of N mass points illustrated in Figure 1. One can apply Newton's second law to the i^{th} mass point in the cloud to obtain

$$\frac{d}{dt}\left(m_i \underline{v}_i\right) = \underline{b}_i + \sum_{j=1}^{j=N} \underline{f}_{ij} \tag{4}$$

Here we have used $\underline{b}_i$ to represent the body force exerted on the i^{th} mass point by the large, spherical body located outside the cloud in Figure 1. The force exerted by the j^{th} mass point in the cloud on the i^{th} mass point in the cloud is represented by $\underline{f}_{ij}$, and this force obeys Newton's third law as indicated by

$$\underline{f}_{ij} = -\underline{f}_{ji} \tag{5}$$

To obtain Newton's second law for the cloud of mass points, we sum Eq. (4) over all the mass points in the cloud[8]

$$\frac{d}{dt}\sum_{i=1}^{i=N} m_i \underline{v}_i = \sum_{i=1}^{i=N} \underline{b}_i + \sum_{i=1}^{i=N}\sum_{j=1}^{j=N} \underline{f}_{ij} \tag{6}$$

and make use of Eq. (5) to simplify this result to the form

$$\frac{d}{dt}\sum_{i=1}^{i=N} m_i \underline{v}_i = \sum_{i=1}^{i=N} \underline{b}_i \tag{7}$$

The mass of the cloud is given by

$$m = \sum_{i=1}^{i=N} m_i \tag{8}$$

while the center of mass, $\underline{r}_{CM}$, and the velocity of the center of mass, $\underline{v}_{CM}$, are defined by

$$\underline{r}_{CM} = \frac{1}{m}\sum_{i=1}^{i=N} m_i \underline{r}_i, \quad \underline{v}_{CM} = \frac{1}{m}\sum_{i=1}^{i=N} m_i \underline{v}_i, \tag{9}$$

The second of these definitions allows us to express Eq. (7) in the form

$$\frac{d}{dt}\left(m\underline{v}_{CM}\right) = \sum_{i=1}^{i=N} \underline{b}_i \tag{10}$$

We now identify the total external force acting on the cloud of mass points as

$$\underline{f} = \sum_{i=1}^{i=N} \underline{b}_i \tag{11}$$

so that Newton's second law for a cloud of mass points is given by

$$\text{Newton II}: \quad \frac{d}{dt}\left(m\underline{v}_{CM}\right) = \underline{f} \tag{12}$$

Feynman, *et al.*,[9] describe this situation by saying "Newton's law has the peculiar property that if it is right on a certain scale [*the mass point scale*], then it will be right on a larger scale [*the cloud scale*]." While this is a satisfying result, it does not explain "how small" a particle must be so that Eq. (2) can be applied with confidence. For rigid bodies the velocity $\underline{v}$ at any point $\underline{r}$ is given by[10]

$$\underline{v}\left(\underline{r}\right) = \underline{v}_{CM} + \underline{\omega} \times \left(\underline{r} - \underline{r}_{CM}\right) \tag{13}$$

in which $\underline{\omega}$ represents the angular velocity. Here we see that a *single velocity* can be used to describe the motion of a rigid body whenever $\underline{\omega}\times(\underline{r} - \underline{r}_{CM})$ is small compared to $\underline{v}_{CM}$, thus the constraint associated with the "mass point" assumption is given by

$$\underline{\omega} \times \left(\underline{r} - \underline{r}_{CM}\right) \ll \underline{v}_{CM} \tag{14}$$

For deformable bodies, one must replace Eq. (13) with the more general representation

$$\underline{v}\left(\underline{r}\right) = \underline{v}_{CM} + \int_{\eta=\underline{r}_{CM}}^{\eta=\underline{r}} \left(\nabla \underline{v}\right)^T d\eta \tag{15}$$

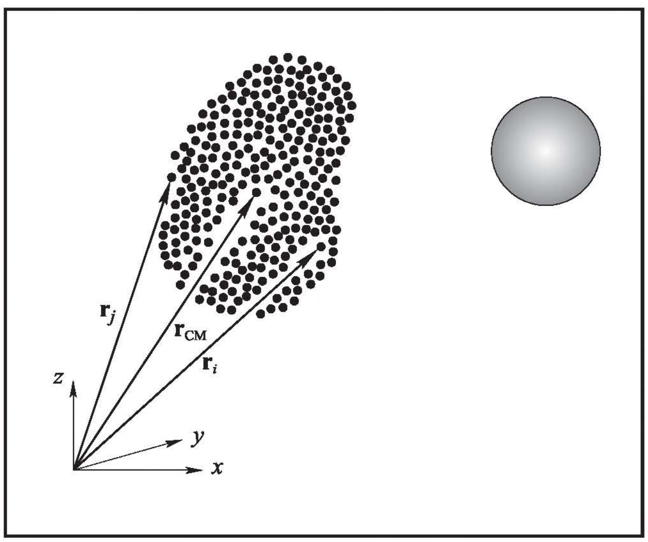


Figure 1. *Cloud of mass points interacting with a body.*

and then examine the velocity gradient tensor in terms of its symmetric and skew-symmetric parts.[11] In this case, the restriction[12] is obviously given by

$$\int_{\eta=\underline{r}_{CM}}^{\eta=\underline{r}} \left(\nabla \underline{v}\right)^T d\eta \ll \underline{v}_{CM} \tag{16}$$

however, the associated constraint would require a more detailed analysis of the fluid deformation. If one accepts Eq. (12) as Newton's second law instead of Eq. (2), no constraint need be imposed.

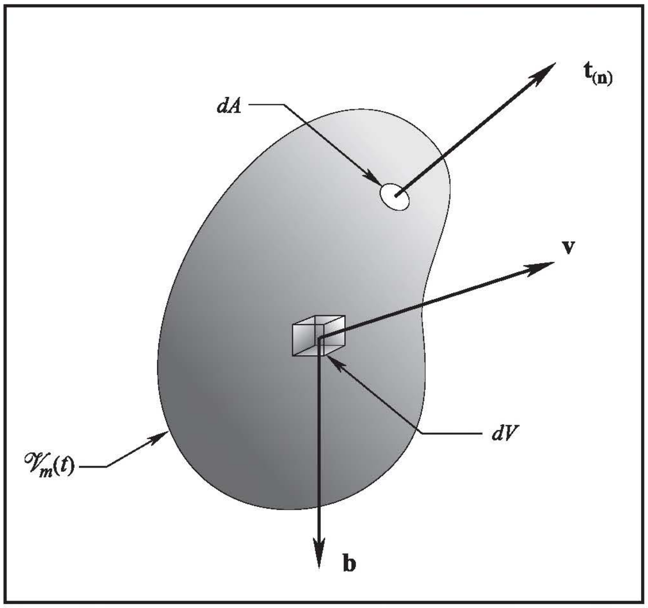


Figure 2. *Moving, deformed body.*

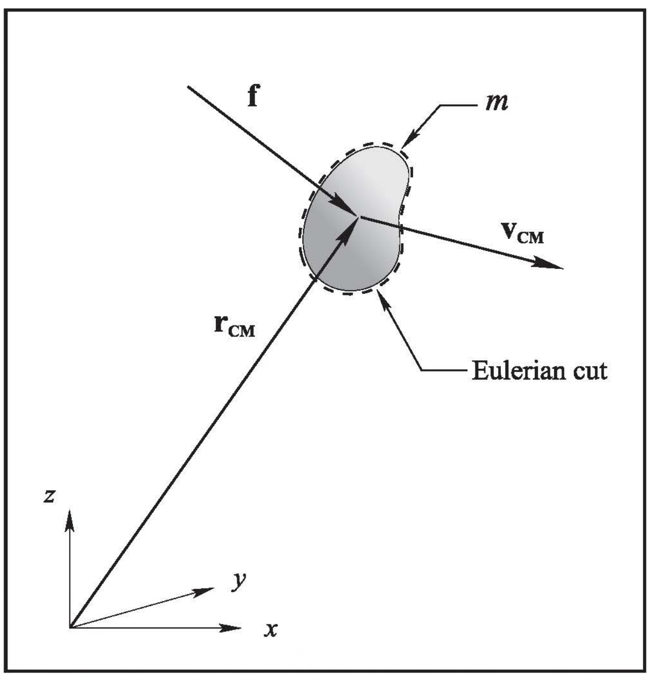


Figure 3. *Motion of a body.*

EULER'S LAWS

While Newton's laws seem to be suitable for the study of mass points and clouds of mass points, they cannot be applied directly to the motion of a moving, deforming, continuous medium.[13] Regardless of what words are used to describe the laws of mechanics used by chemical engineers, those laws are indeed the laws proposed by Euler that can be stated as

Euler (1707-1783)

> I. *The time rate of change of the momentum of a body equals the force acting on the body.*
>
> II. *The time rate of change of the angular momentum of a body equals the torque acting on the body, where both the torque and the moment are taken with respect to the same fixed point.*

In addition to these two laws, we accept the *Euler cut principle*[14] that can be stated as:

> *Not only do the laws of continuum physics apply to* distinct bodies *but they also apply to any arbitrary body that one might imagine as being* cut out of *a distinct body.*

To understand how Euler's laws are related to Newton's laws, we need to express Euler's laws in precise mathematical form. This will allow us to demonstrate that they contain Newton's laws provided that we restrict ourselves to *non-relativistic phenomena.*

For the body occupying the *material volume*, $\mathscr{V}_m(t)$, illustrated in Figure 2, Euler's laws are given by[15]

$$\text{Euler I:} \quad \frac{d}{dt}\int_{\mathscr{V}_m(t)} \rho \underline{v}\, dV = \int_{\mathscr{V}_m(t)} \rho \underline{b}\, dV + \int_{\mathscr{A}_m(t)} \underline{t}_{(\underline{n})}\, dA \tag{17}$$

$$\text{Euler II:} \quad \frac{d}{dt}\int_{\mathscr{V}_m(t)} \underline{r}\times\rho \underline{v}\, dV = \int_{\mathscr{V}_m(t)} \underline{r}\times\rho \underline{b}\, dV + \int_{\mathscr{A}_m(t)} \underline{r}\times\underline{t}_{(\underline{n})}\, dA \tag{18}$$

To be clear about Euler's two laws, we need to say that the velocity, $\underline{v}$, is determined relative to an *inertial frame* and that the position vector $\underline{r}$ is determined relative to some fixed point in an inertial frame. As mentioned earlier in connection with Newton's laws, one identifies an inertial frame by experiment. It is important to remember that these two axiomatic statements for *linear momentum* and *angular momentum* apply to any *arbitrary body* that one imagines as being cut out of a distinct body.

EULER'S LAWS AND NEWTON'S LAWS

Given Euler's two laws of mechanics and the Euler cut principle, we need to know how they are related to Newton's three laws. To explore this problem, we consider a body of mass m illustrated in Figure 3, and we locate the center of mass of that body in terms of the position vector defined by

$$\underline{r}_{CM} = \frac{1}{m}\int_{\mathscr{V}_m(t)} \rho \underline{r}\, dV \tag{19}$$

For a sphere of uniform density, the center of mass would be located at the geometrical center of the sphere; however, the definition of $\underline{r}_{CM}$ is completely general and Eq. (19) is applicable to any arbitrary body that is cut out of a distinct body. The velocity of the center of mass is defined in a similar manner

$$\underline{v}_{CM} = \frac{1}{m} \int_{\mathcal{V}_m(t)} \rho \underline{v} dV \tag{20}$$

and one can use a special form of the Reynolds transport theorem[16] to prove that

$$\underline{v}_{CM} = \frac{d\underline{r}_{CM}}{dt} \tag{21}$$

The definition given by Eq. (20) can be used to express the first term in Eq. (17) as

$$\frac{d}{dt} \int_{\mathcal{V}_m(t)} \rho \underline{v} dV = \frac{d}{dt}\left(m\underline{v}_{CM}\right) \tag{22}$$

As a matter of convenience, we designate the total force acting on the body by

$$\underline{f} = \int_{\mathcal{V}_m(t)} \rho \underline{b} dV + \int_{\mathcal{A}_m(t)} \underline{t}_{(\underline{n})} dA \tag{23}$$

so that Eq. (17) can be represented in the simplified form given by

$$\text{Euler Result I:} \qquad \frac{d}{dt}\left(m\underline{v}_{CM}\right) = \underline{f} \tag{24}$$

This is identical in form to Newton's second law for the cloud of mass points illustrated in Figure 1, and if the body is "small enough" so that $\underline{v}_{CM}$ can be replaced by $\underline{v}$ we see that Eq. (24) is identical in form to Eq. (2) for a mass point. The similarity in form (*not content*) of Euler's first law and Newton's second law has encouraged many to think that Newton's laws and Euler's first law are essentially equivalent. This is a line of thought that should be discouraged.

BODY FORCES

To clarify the different perspectives of physicists and chemical engineers, we apply Euler's first and second laws to the special case of three interacting bodies in a vacuum. This situation is illustrated in Figure 4 where we have shown two distinct small bodies, three Eulerian cuts (material volumes), and a distinct large body. For Cut I and Cut II, Euler's first law yields

$$\text{Cut I:} \quad \frac{d}{dt} \int_{\mathcal{V}_I(t)} \rho_1 \underline{v}_1 dV = \int_{\mathcal{V}_I(t)} \rho_1 \underline{b}_{12} dV + \int_{\mathcal{V}_I(t)} \rho_1 \underline{b}_{13} dV \tag{25}$$

$$\text{Cut II:} \quad \frac{d}{dt} \int_{\mathcal{V}_{II}(t)} \rho_2 \underline{v}_2 dV = \int_{\mathcal{V}_{II}(t)} \rho_2 \underline{b}_{21} dV + \int_{\mathcal{V}_{II}(t)} \rho_2 \underline{b}_{23} dV \tag{26}$$

The application of Cut III treats the two small bodies as a single body for which the time rate of change of momentum is balanced by the applied external force. This leads to

$$\text{Cut III:} \quad \frac{d}{dt}\left[\int_{\mathcal{V}_I(t)} \rho_1 \underline{v}_1 dV + \int_{\mathcal{V}_{II}(t)} \rho_2 \underline{v}_2 dV\right] = \int_{\mathcal{V}_I(t)} \rho_1 \underline{b}_{13} dV + \int_{\mathcal{V}_{II}(t)} \rho_2 \underline{b}_{23} dV \tag{27}$$

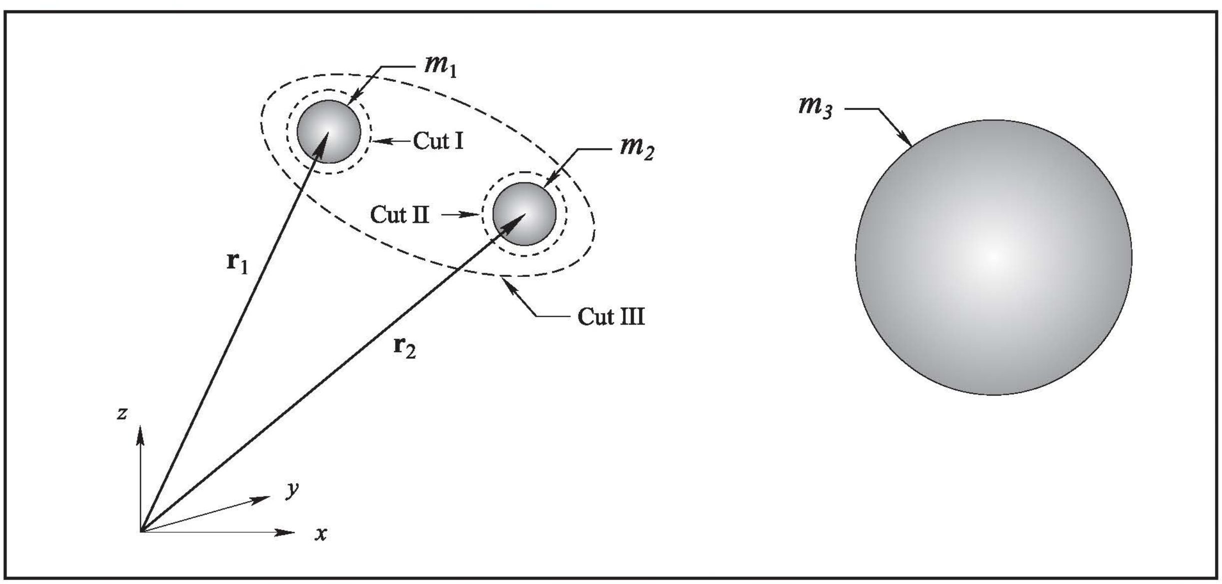


Figure 4. *Three-body process.*

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Substitution of Eqs. (25) and (26) into Eq. (27) leads to

$$\int_{\mathscr{V}_I(t)} \rho_1 \underline{b}_{12} dV + \int_{\mathscr{V}_{II}(t)} \rho_2 \underline{b}_{21} dV = 0 \qquad (28)$$

and it will be convenient to identify these two body forces as

$$\underline{f}_{12} = \int_{\mathscr{V}_I(t)} \rho_1 \underline{b}_{12} dV, \quad \underline{f}_{21} = \int_{\mathscr{V}_{II}(t)} \rho_2 \underline{b}_{21} dV \qquad (29)$$

At this point we repeat Eq. (24) as

$$\text{Euler Result I:} \quad \frac{d}{dt}\left(m\underline{v}_{CM}\right) = \underline{f} \qquad (30)$$

And note that Eqs. (28) and (29) lead to

$$\text{Euler Result II:} \quad \underline{f}_{12} = -\underline{f}_{21} \qquad (31)$$

Eq. (30) yields Newton's second law for the cloud of mass points illustrated in Figure 3, and if Eq. (30) is applied to a single mass point it yields Newton's second law as given by Eq. (2). Eq. (31), which was derived by applying Euler's first law to the process illustrated in Figure 4, is identical to Newton's third law. Here we see that Euler's first law can be used to obtain *all three of Newton's laws*; however, the inverse is not true, *i.e.*, one cannot use Newton's laws for mass points or for a cloud of mass points to obtain Euler's first law. Euler's laws are based on the Euler cut principle and the assumption that the material under consideration can be treated as a continuum, and these constructs are not to be found in Newton's treatment of mechanics.[17]

Given that Euler's first law contains *all that is available in Newton's three laws*, one must wonder why physicists do not move forward one century and accept Euler's first law as their axiom for mechanics. The answer would appear to be associated with Euler's second law that we examine in the following paragraphs.

CENTRAL FORCES

In the absence of any surface forces, we can express Euler's second law as

$$\frac{d}{dt}\int_{\mathscr{V}_m(t)} \underline{r} \times \rho \underline{v} dV = \int_{\mathscr{V}_m(t)} \underline{r} \times \rho \underline{b} dV \qquad (32)$$

and for the three Eulerian cuts, or *control volumes*, illustrated in Figure 4 we have

$$\text{Cut I:} \quad \frac{d}{dt}\int_{\mathscr{V}_I(t)} \underline{r}_1 \times \rho_1 \underline{v}_1 dV = \int_{\mathscr{V}_I(t)} \underline{r}_1 \times \rho_1 \underline{b}_{12} dV + \int_{\mathscr{V}_I(t)} \underline{r}_1 \times \rho_1 \underline{b}_{13} dV \qquad (33)$$

$$\text{Cut II:} \quad \frac{d}{dt}\int_{\mathscr{V}_{II}(t)} \underline{r}_2 \times \rho_2 \underline{v}_2 dV = \int_{\mathscr{V}_{II}(t)} \underline{r}_2 \times \rho_2 \underline{b}_{21} dV + \int_{\mathscr{V}_{II}(t)} \underline{r}_2 \times \rho_2 \underline{b}_{23} dV \qquad (34)$$

$$\text{Cut III:} \quad \frac{d}{dt}\int_{\mathscr{V}_I(t)} \underline{r}_1 \times \rho_1 \underline{v}_1 dV + \frac{d}{dt}\int_{\mathscr{V}_{II}(t)} \underline{r}_2 \times \rho_2 \underline{v}_2 dV$$

$$= \int_{\mathscr{V}_I(t)} \underline{r}_1 \times \rho_1 \underline{b}_{13} dV + \int_{\mathscr{V}_{II}(t)} \underline{r}_2 \times \rho_2 \underline{b}_{23} dV \qquad (35)$$

Use of Eqs. (33) and (34) in Eq. (35) leads to a constraint on the body forces given by

$$\int_{\mathscr{V}_I(t)} \underline{r}_1 \times \rho_1 \underline{b}_{12} dV + \int_{\mathscr{V}_{II}(t)} \underline{r}_2 \times \rho_2 \underline{b}_{21} dV = 0 \qquad (36)$$

The position vectors can be expressed in terms of the position vectors locating the centers of mass according to

$$\underline{r}_1 = \left(\underline{r}_{CM}\right)_1 + \underline{\tilde{r}}_1, \quad \underline{r}_2 = \left(\underline{r}_{CM}\right)_2 + \underline{\tilde{r}}_2 \qquad (37)$$

and this leads to

$$\left(\underline{r}_{CM}\right)_1 \times \int_{\mathscr{V}_I(t)} \rho_1 \underline{b}_{12} dV + \int_{\mathscr{V}_I(t)} \underline{\tilde{r}}_1 \times \rho_1 \underline{b}_{12} dV$$

$$+ \left(\underline{r}_{CM}\right)_2 \times \int_{\mathscr{V}_{II}(t)} \rho_2 \underline{b}_{21} dV + \int_{\mathscr{V}_{II}(t)} \underline{\tilde{r}}_2 \times \rho_2 \underline{b}_{21} dV = 0 \qquad (38)$$

Next we make use of Eqs. (28) and (29) to express this result in the form

$$\left[\left(\underline{r}_{CM}\right)_1 - \left(\underline{r}_{CM}\right)_2\right] \times \underline{f}_{12} + \left[\int_{\mathscr{V}_I(t)} \underline{\tilde{r}}_1 \times \rho_1 \underline{b}_{12} dV + \int_{\mathscr{V}_{II}(t)} \underline{\tilde{r}}_2 \times \rho_2 \underline{b}_{21} dV\right] = 0 \qquad (39)$$

In the appendix we demonstrate that the last term in this result can be neglected when the following constraint is satisfied:

$$\frac{O(\tilde{\underline{r}}_1)+O(\tilde{\underline{r}}_2)}{O\left[(\underline{r}_{CM})_1-(\underline{r}_{CM})_2\right]} \ll 1 \qquad (40)$$

Under these circumstances, Euler's second law leads to

$$\left[(\underline{r}_{CM})_1-(\underline{r}_{CM})_2\right]\times \underline{f}_{12}=0 \qquad (41)$$

and there are three ways in which this result can be satisfied. We list the three possibilities as

1. $(\underline{r}_{CM})_1-(\underline{r}_{CM})_2=0$ (42)
2. $\underline{f}_{12}=0$ (43)
3. $(\underline{r}_{CM})_1-(\underline{r}_{CM})_2$ and f_{12} are parallel (44)

Since the first two possibilities can not be generally true, we conclude that the interaction force between two bodies must be *parallel* to the vector $(\underline{r}_{CM})_1 - (\underline{r}_{CM})_2$. We express this result as

Euler Result III: $$\underline{f}_{12}=\Omega_{12}\left[(\underline{r}_{CM})_1-(\underline{r}_{CM})_2\right] \qquad (45)$$

in which Ω_{12} is some *scalar parameter* of the interaction force law. Eq. (45) indicates that the interaction force between two bodies subject to the constraint given by Eq. (40) must act along the line of centers, *i.e.*, it is a *central force*.

In this analysis we have shown that Euler's *first law* contains Newton's *three laws*, while Euler's *second law* provides what is known as the *central force law* for the case of mass-point mechanics. Given the power and economy of Euler's laws, one can wonder why Newton's *three laws* are not discarded in favor of Euler's *two laws*. The answer lies in the fact that the *central force law*, given by Eq. (45), represents a *non-relativistic phenomenon*. Since forces are propagated at the speed of light, the force that one body exerts on another *cannot lie along the line of centers* when the relative velocity between the two bodies approaches the speed of light. Because of this, physicists prefer to view mechanical phenomena in terms of Newton's laws and make use of the central force law as a *special case* that can be discarded when relativistic phenomena are encountered. Engineers, on the other hand, are rarely involved in relativistic phenomena and what is a *special case* for the physicist is the *general case* for the engineer. Because of this, engineers uniformly formulate their mechanical problems in terms of Euler's two laws and the Euler cut principle.

CONCLUSIONS

Physicists, who begin teaching chemical engineering students about the laws of mechanics, are committed to a Newtonian perspective because it is consistent with relativistic mechanics and mass points. This perspective will not change.

Physicists prefer to view mechanical phenomena in terms of Newton's laws and make use of the central force law as a special case that can be discarded when relativistic phenomena are encountered. Engineers, on the other hand, are rarely involved in relativistic phenomena and what is a* special case *for the physicist is the* general case *for the engineer.

Chemical engineering faculty teach chemical engineering students about Euler's laws of mechanics, regardless of what words they use to describe these laws. Chemical engineering faculty need to take responsibility for the development of a smooth transition between the perspective of physicists and the perspective of engineers. In the absence of such a smooth transition, our students will be confronted with a discontinuity[18] and will never be completely confident in the laws of mechanics that they have been given.

ACKNOWLEDGMENT

The author would like to thank the reviewers for thoughtful and helpful comments.

NOMENCLATURE

- $\mathscr{A}_m(t)$ surface area of a material volume, m^2
- $\underline{b}$ total body force per unit mass, N/kg
- $\underline{b}_i$, i=1,2,...,N, body force exerted by a large, external body on the i^{th} mass point, N
- $\underline{b}_{12}$ body force per unit mass exerted by body #2 on body #1, N/kg
- $\underline{b}_{21}$ body force per unit mass exerted by body #1 on body #2, N/kg
- $\underline{f}$ force, N
- $\underline{f}_{12}$ force exerted by body #2 on body #1, N/kg
- $\underline{f}_{21}$ force exerted by body #1 on body #2, N/kg
- $\underline{f}_{ij}$ force exerted by the j^{th} mass point on the i^{th} mass point in a cloud of mass points, N
- m mass of a body or mass of a cloud of mass points, kg
- m_i mass of the i^{th} mass point, kg
- $\underline{n}$ unit normal vector
- $\underline{r}$ position vector, m
- $\underline{r}_{CM}$ position vector locating the center of mass, m
- t time, s
- $\underline{t}_{(n)}$ stress vector, N/m^2

$\underline{v}_i$ velocity of the i^{th} mass point, m/s
$\underline{v}$ velocity, m/s
$\underline{v}_{CM}$ velocity of the center of mass, m/s
$\mathscr{V}_m(t)$ volume of a body (material control volume), m^3

Greek Letters

ρ total mass density, kg/m^3
ρ_i total mass density of the i^{th} body, kg/m^3
$\underline{\omega}$ angular velocity, rad/s

REFERENCES

1. Truesdell, C., *Essays in the History of Mechanics*, Springer-Verlag, New York, p.88 (1968)
2. Truesdell, C., *Essays in the History of Mechanics*, Springer-Verlag, New York, p. 167 (1968)
3. Feynman, R.P., R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley Publishing Company, New York, **I**, p. 9-1 (1963)
4. Huggins, E.R., *Physics 1*, W.A. Benjamin, Inc., New York, p. 109 (1968)
5. Greider, K., *Invitation to Physics*, Harcourt Brace Jovanovich, Inc., New York, p. 38 (1973)
6. Hurley, J.P., and C. Garrod, *Principles of Physics*, Houghton Mifflin Co., Boston, p. 49 (1978)
7. Feynman, R.P., R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley Publishing Company, New York, **I**, p.18-1 (1963)
8. Marion, J.B., *Classical Dynamics of Particles and Systems*, Academic Press, New York, p. 68 (1970)
9. Feynman, R.P., R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison-Wesley Publishing Company, New York, **I**, p. 19-2 (1963)
10. Landau, L.D., and E.M. Lifshitz, *Mechanics*, Pergamon Press, New York (1960)
11. Aris, R., V*ectors, Tensors, and the Basic Equations of Fluid Mechanics*, Prentice-Hall, Inc., Englewood Cliffs, NJ, p. 89 (1962)
12. Whitaker, S., "Levels of Simplification: The Use of Assumptions, Restrictions and Constraints in Engineering Analysis," *Chem. Eng. Educ.*, **22**, 104 (1988)
13. Serrin, J., "Mathematical Principles of Classical Fluid Mechanics," in *Handbuch der Physik*, Vol. VIII, Part 1, edited by S. Flugge and C. Truesdell, Springer Verlag, New York, page 134 (1959)
14. Truesdell, C., *Essays in the History of Mechanics*, Springer-Verlag, New York, page 193 (1968)
15. Whitaker, S., *Introduction to Fluid Mechanics*, R.E. Krieger Pub. Co., Malabar, FL (1981)
16. Aris, R., V*ectors, Tensors, and the Basic Equations of Fluid Mechanics*, Prentice- Hall, Inc., Englewood Cliffs, NJ, p. 87 (1962)
17. Truesdell, C., "A Program Toward Rediscovering the Rational Mechanics of the Age of Reason," in *Essays in the History of Mechanics*, Springer-Verlag, New York (1968)
18. Whitaker, S., "Discontinuities in Chemical Engineering Education, *Chem. Eng. Educ.*, **33**, 18 (1999)
19. Birkhoff, G., *Hydrodynamics: A Study in Logic, Fact, and Similitude*, Princeton University Press, Princeton, NJ, p. 4 (1960)
20. Stein, S.K., and A. Barcellos, *Calculus and Analytic Geometry*, McGraw-Hill, Inc., New York, p. 691 (1992)

APPENDIX: CENTRAL FORCE LAW

Deciding when some quantity is "small enough" so that it can be discarded is not always an easy task. Here we consider the simplification that led from Eq. (39) to (40) and the central force law represented by Eq. (45). We begin with Eq. (A1)

$$\left[\left(\underline{r}_{CM}\right)_1-\left(\underline{r}_{CM}\right)_2\right]\times\underline{f}_{12}+\left[\int_{\mathscr{V}_I(t)}\tilde{\underline{r}}_1\times\rho_1\underline{b}_{12}dV+\int_{\mathscr{V}_{II}(t)}\tilde{\underline{r}}_2\times\rho_2\underline{b}_{21}dV\right]=0 \quad (A1)$$

and use the following nomenclature

$$\left[\left(\underline{r}_{CM}\right)_1-\left(\underline{r}_{CM}\right)_2\right]=\underline{R} \quad (A2a)$$

$$\underline{f}_{12}=\underline{F} \quad (A2b)$$

$$\int_{\mathscr{V}_I(t)}\tilde{\underline{r}}_1\times\rho_1\underline{b}_{12}dV+\int_{\mathscr{V}_{II}(t)}\tilde{\underline{r}}_2\times\rho_2\underline{b}_{21}dV=D \quad (A2c)$$

to express Eq. (A1) as

$$\underline{R}\times\underline{F}+\underline{D}=0 \quad (A3)$$

Here we would like to know when the vector $\underline{D}$ can be discarded in order to simplify this result. A plausible intuitive hypothesis[19] associated with this simplification is given by

Assumption: $$\underline{R}\times\underline{F}=0 \quad (A4)$$

however, we cannot discard $\underline{D}$ as being small compared to $\underline{R}\times\underline{F}$ since Eq. (A3) requires that $\underline{D}$ and $\underline{R}\times\underline{F}$ be the *same order of magnitude*. This type of problem has been considered before,[12] and we will follow the procedure suggested in that earlier work. This requires that we *decompose* $\underline{F}$ into a part that is parallel to $\underline{R}$ and a part that is perpendicular to $\underline{R}$ as indicated by[20]

$$\underline{F}=\underbrace{\underline{F}_{=}}_{\text{parallel part}}+\underbrace{\underline{F}_{\perp}}_{\text{perpendicular part}} \quad (A5)$$

On the basis of this decomposition, we see that Eq. (A3) provides the two results given by

$$\underline{R}\times\underline{F}_{=}=0 \quad (A6a)$$

$$\underline{R}\times\underline{F}_{\perp}+\underline{D}=0 \quad (A6b)$$

This allows us to *estimate* as $\underline{F}_{\perp}$

$$\underline{F}_{\perp}=\frac{O(\underline{D})}{O(\underline{R})} \quad (A7)$$

in which O indicates an order of magnitude estimate. If $\underline{F}_{\perp}$ is small relative to $\underline{F}_{=}$, and if small causes give rise to small effects, we can replace $\underline{F}_{=}$ with F and Eq. (A6a) leads to the central force law given by Eq. (45). To develop the conditions that must be satisfied in order that $\underline{F}_{\perp}$ be negligible compared to $\underline{F}_{=}$, we consider the inequality given by

$$\underline{F}_{=}\gg\underline{F}_{\perp} \quad (A8)$$

In terms of the estimate given by Eq. (A7) this leads to

$$\underline{\underline{F}} \gg \frac{O(\underline{D})}{O(\underline{R})} \tag{A9}$$

and because of the constraint given by Eq. (A8) we can express this result as

$$\underline{F} \gg \frac{O(\underline{D})}{O(\underline{R})} \tag{A10}$$

Making use of the definitions given by Eqs. (A2) this inequality can be arranged in the form

$$\frac{O\left[\int_{\mathscr{V}_I(t)} \underline{\tilde{r}}_1 \times \rho_1 \underline{b}_{12} dV + \int_{\mathscr{V}_{II}(t)} \underline{\tilde{r}}_2 \times \rho_2 \underline{b}_{21} dV\right]}{O(\underline{f}_{12}) O\left(\left[(\underline{r}_{CM})_1 - (\underline{r}_{CM})_2\right]\right)} \ll 1 \tag{A11}$$

On the basis of Eqs. (29) we obtain the two estimates

$$\int_{\mathscr{V}_I(t)} \underline{\tilde{r}}_1 \times \rho_1 \underline{b}_{12} dV = O(\underline{\tilde{r}}_1) \underline{f}_{12}, \quad \int_{\mathscr{V}_{II}(t)} \underline{\tilde{r}}_2 \times \rho_2 \underline{b}_{21} dV = O(\underline{\tilde{r}}_2) \underline{f}_{21} \tag{A12}$$

and use of these [along with Eq. (31)] in Eq. (A11) leads to the constraint given by

$$\text{Constraint:} \qquad \frac{O(\underline{\tilde{r}}_1) + O(\underline{\tilde{r}}_2)}{O\left[(\underline{r}_{CM})_1 - (\underline{r}_{CM})_2\right]} \ll 1 \tag{A13}$$

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