LEVEL CONTROL BY REGULATING CONTROL VALVE AT THE BOTTOM OF A GRAVITY-DRAINED TANK

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Typical liquid-level control systems used in industry are illustrated in Figures 1 (next page). The majority of process control textbooks cover control systems as shown in Figure 1(a) and (b). In the Unit Operations Laboratory of the author's department, there is a level-control experiment with a direct-acting (or fail-close) control valve located at the bottom of the tank [Figure 1(c)]. Therefore, the author developed an instructional module that provides a thorough analysis of the dynamic behavior of the control system in Figure 1 (c). Transfer functions for the controller output and the process variable in a feedback proportional-integral (PI) control system are derived for the servo problem (setpoint tracking) and the regulatory problem (disturbance rejection).

One main challenge to the development of dynamic models for the case illustrated in Figure 1(c) emanates from the fact that the discharge flow rate is governed not only by the liquid level (h, the process or controlled variable) but also by the size, type, and valve stem position (x, the manipulated variable) of the control valve. The following assumptions are made in the derivation of the dynamic models: (1) the head loss of liquid in the discharge line is entirely due to the control valve; (2) the control valve is of the direct-acting (or fail-close) type and the valve trim is linear, *i.e.*, the valve characteristic function $C_v(x)$ is linearly proportional to x; (3) the valve is never saturated (fully closed or fully open) during the dynamic response, and; (4) the flow rate of liquid through the control valve is proportional to $(\Delta P_{valve})^{1/2}$, where ΔP_{valve} is the pressure drop across the valve which can be taken as (ρ g h) based on assumption one above, where ρ is the liquid density and g the gravitational acceleration. We may then express the discharge volumetric flow rate q [m^3/s] as

$$q=(x,h)=C_v(x)\sqrt{\frac{\Delta P_{valve}}{S.G.}}=kx\sqrt{h} \qquad (1)$$

where x is the valve stem position or the extent of valve opening, with x = 0 being fully closed and x = 1 fully open for a direct-acting (or fail-close) control valve; S.G. is the specific gravity of the liquid, h is the liquid level measured from the bottom of the tank, and k is a lumped constant.

The initial steady-state volumetric balance (input flow rate = output flow rate) for this constant-density system is

$$A\frac{dh_{ss}}{dt}=0=q_{in,ss}-q_{ss} \qquad (2)$$

where t is time, A is the cross-sectional area of the open tank, subscript ss denotes the initial steady-state condition or null operating condition, and q_{in} is the incoming volumetric flow rate. Assuming that at t = 0 both q_{in} and q start to deviate from their respective steady-state values, the response of the liquid level can be described by a transient-state balance equation:

$$A\frac{dh}{dt}=q_{in}-q \qquad (3)$$

When Eq. (2) is subtracted from Eq. (3), we have a differential equation whose variables are expressed as deviation quantities:

$$A\frac{d\Delta h}{dt}=\Delta q_{in}-\Delta q \qquad (4)$$

where

$$\Delta h=h-h_{ss} \qquad (5)$$

$$\Delta q_{in}=q_{in}-q_{in,ss} \qquad (6)$$

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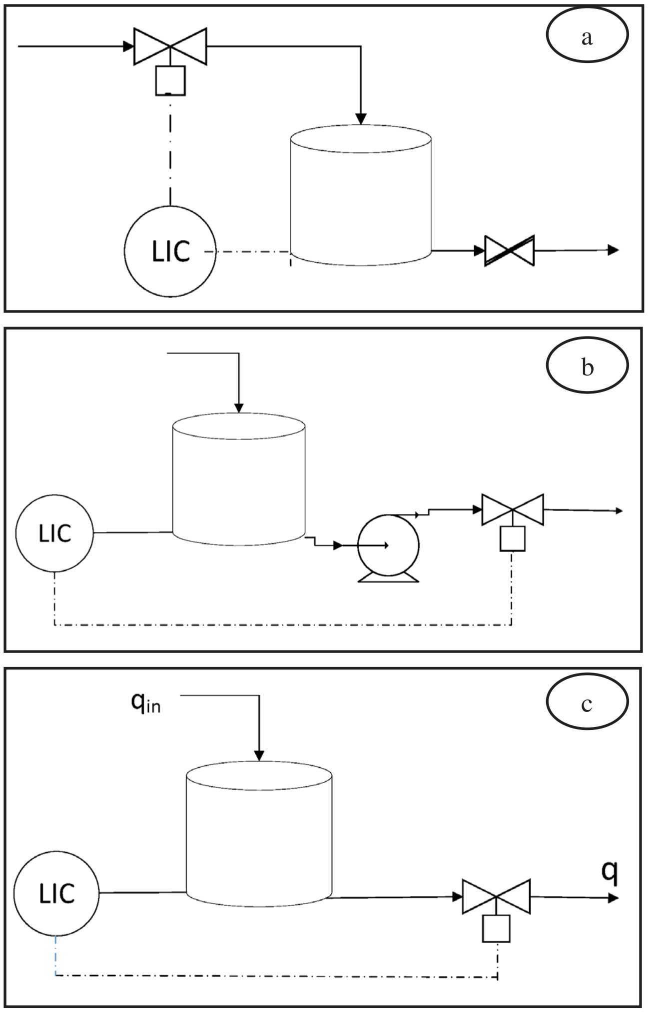
ChE errata

Due to a production error, there is one symbol missing in the paper "Level Control..." by Larry K. Jang, published in the Fall 2016 issue of *CEE*. The letter "f" is missing in the final print. Eq. (1) on Page 245 should appear like

$$q=f(x,h)=C_v(x)\sqrt{\frac{\Delta P_{valve}}{S.G.}}=kx\sqrt{h} \qquad (1)$$



Figures 1. Typical liquid-level control systems.

It is noted that a nonlinear term $[\Delta q = q - q_{ss} = kx(h)^{1/2} - kx_{ss}(h_{ss})^{1/2}]$ is encountered, which needs to be linearized:

$$\begin{aligned}\Delta q &= q - q_{ss} \\ &= f(x,h) - f(x_{ss}, h_{ss}) \\ &\approx \left(\frac{\partial f}{\partial x}\right)_{ss}\Delta x + \left(\frac{\partial f}{\partial h}\right)_{ss}\Delta h \\ &= k\sqrt{h_{ss}}\Delta x + \frac{k\,x_{ss}}{2\sqrt{h_{ss}}}\Delta h \end{aligned} \tag{7}$$

where

$$\Delta x = x - x_{ss} \tag{8}$$

If both sides of the resultant linearized differential equation [Eqs. (4) and (7) combined] are multiplied with

$$\frac{2\sqrt{h_{ss}}}{k\,x_{ss}},$$

we have a first-order differential equation in the standard form with the coefficient of the process variable Δh being 1:

$$\tau_p \frac{d\Delta h}{dt} = R\Delta q_{in} + K_p\Delta x - \Delta h \tag{9}$$

where

$$R = \frac{2\sqrt{h_{ss}}}{k\,x_{ss}} \tag{10}$$

$$K_p = \text{Process Gain} = -\frac{2h_{ss}}{x_{ss}} \tag{11}$$

$$\begin{aligned}\tau_p &= \text{first-order time constant} \\ &= AR\end{aligned} \tag{12}$$

By applying a Laplace transform with the initial condition $\Delta h(t = 0) = 0$, we arrive at the open-loop transfer function for the liquid level:

$$\Delta h(s) = \frac{R}{\tau_p s + 1}\Delta q_{in}(s) + \frac{K_p}{\tau_p s + 1}\Delta x(s) \tag{13}$$

For simplicity, the symbol "(s)" that denotes the Laplace domain following each variable is omitted hereafter. In this process, Δq_{in} is considered the load or disturbance and Δx the manipulated variable. Therefore, we may define the process transfer function G_p and the disturbance transfer function G_d in the Laplace domain:

$$G_p = \frac{K_p}{\tau_p s + 1} \tag{14}$$

$$\begin{aligned}G_d &= \frac{K_d}{\tau_p s + 1} \\ &= \frac{R}{\tau_p s + 1}\end{aligned} \tag{15}$$

where the disturbance gain $K_d = R$ and the disturbance time constant is identical to the process time constant τ_p. We may then express the feedback control system as a block diagram shown in Figure 2, assuming that the dynamics of both the actuator and the sensor are tentatively ignored and that the actuator gain is lumped together with the controller gain.

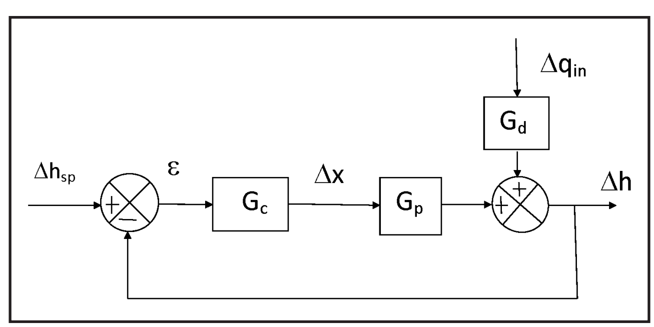


***Figure 2.** Feedback control block diagram for the liquid-level control system in Figure 1(c).*

TRANSFER FUNCTIONS FOR Δx, Δh, AND Δq IN THE SETPOINT TRACKING CASE

Assume that a proportional-integral (PI) controller is used to adjust the control valve stem position according to error ε and the transfer function G_c for the controller:

$$G_c = \frac{\Delta x}{\varepsilon} = K_c\left(1+\frac{1}{\tau_I s}\right) \tag{16}$$

where

K_c = Proportional Gain of the PI Controller

τ_I = Integral Time of the PI Controller

$\varepsilon = \text{Error} = h_{sp} - h = \Delta h_{sp} - \Delta h$

The subscript sp denotes the level setpoint.

Transfer Function for the Liquid Level

With the process model G_p defined, the transfer function for the closed-loop setpoint-tracking case $\Delta h/\Delta h_{sp}$ using a PI controller can be found in the literature [1(a),2]

$$\frac{\Delta h}{\Delta h_{sp}} = \frac{G_c G_p}{1+G_c G_p} \tag{17}$$

$$\frac{\Delta h}{\Delta h_{sp}} = \frac{\tau_I s+1}{\tau^2 s^2 + 2\zeta\tau s+1} \tag{18}$$

where

$$\tau = \sqrt{\frac{\tau_p \tau_I}{K_c K_p}} \tag{19}$$

$$\zeta = \frac{1+K_c K_p}{2}\sqrt{\frac{\tau_I}{\tau_p K_c K_p}} \tag{20}$$

Therefore, if a step change in Δh_{sp} with the magnitude **a** is made,

$$\Delta h_{sp} = \frac{a}{s} \tag{21}$$

Then Eq. (18) becomes

$$\Delta h = \frac{a\,\tau_I}{\tau^2 s^2 + 2\zeta\tau s+1} + \frac{a}{s\left(\tau^2 s^2 + 2\zeta\tau s+1\right)}$$
$$= a\,\tau_I Y_{imp} + a Y_{step} \tag{22}$$

where Y_{imp} is the response of a standard second-order model (with gain K = 1) to the unit impulse input and Y_{step} is that to the unit step input. The time-domain equations for Y_{imp} and Y_{step} can be found in the literature.[2,3]

Transfer Function for Control Valve Stem Position

Most textbooks do not provide derivations for the transfer function of the manipulated variable, which is the control valve stem position x in this case. The author derived the transfer function for the control valve stem position x in this instruction module and asked the students to perform their own detailed derivations as a homework exercise. The detailed derivation for the resultant equations below is available to any interested readers upon request. In the setpoint-tracking case, $\Delta q_{in} = 0$. So, $\Delta x\, G_p = \Delta h$ and $\Delta x/\Delta h_{sp} = \Delta h/(G_p\, \Delta h_{sp})$. Therefore, by dividing Eq. (17) with G_p, we have

$$\frac{\Delta x}{\Delta h_{sp}} = \frac{G_c}{1+G_c G_p}$$
$$= K_c + \frac{\beta s}{\tau^2 s^2 + 2\zeta\tau s+1} + \frac{\gamma}{\tau^2 s^2 + 2\zeta\tau s+1} \tag{23}$$

where

$$\beta = \left(\frac{\tau_p - K_c K_p \tau_I}{K_p}\right) \tag{24}$$

$$\gamma = \left(\frac{1-K_c K_p}{K_p}\right) \tag{25}$$

Therefore, if a step change in Δh_{sp} with the magnitude **a** is made (Δh_{sp} = **a**/s), the resultant transfer function for the valve stem position is

$$\Delta x = \frac{K_c a}{s} + \beta a Y_{imp} + \gamma\, a\, Y_{step} \tag{26}$$

The time-domain equation Δx(t) is the sum of a step change with the magnitude K_c **a**, the response of a second-order model to an impulse input ($\beta \mathbf{a} Y_{imp}$), and the response of a second-order model to a step input ($\gamma \mathbf{a} Y_{step}$), where the standard functions Y_{imp} and Y_{step} in the time domain are available in the literature.[2,3] Once the transfer functions and the time-domain equations for Δh and Δx are obtained, the change in discharge flow rate Δq is then expressed as a linear combination as shown in Eq. (7).

TRANSFER FUNCTIONS FOR Δx, Δh, AND Δq IN THE DISTURBANCE REJECTION CASE

Transfer Function for the Liquid Level

The response of the liquid level to a change in disturbance variable (incoming flow rate q_{in}) in a closed-loop PI-control system can be found in the literature.[1(b)]

$$\frac{\Delta h}{\Delta q_{in}} = \frac{G_d}{1+G_c G_p}$$
$$= \frac{\left(\frac{K_d \tau_I}{K_c K_p}\right)s}{\tau^2 s^2 + 2\zeta\tau s+1} \tag{27}$$

Therefore, for a step change in the disturbance variable with the magnitude **b** (Δq_{in} = **b**/s), the transfer function for Δh is

$$\Delta h = \frac{\left(\frac{K_d \tau_I}{K_c K_p}\right) b}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$= \left(\frac{K_d \tau_I}{K_c K_p}\right) b\, Y_{imp} \quad (28)$$

Transfer Function for Control Valve Stem Position

Most textbooks do not provide derivations for the transfer function of the manipulated variable in the disturbance rejection case. Again, the author derived it for the instruction manual and asked students to perform their own derivations as a homework exercise. The author would like to make the detailed derivation for the resultant equations below available to any interested readers upon request. In the disturbance rejection case ($\Delta h_{sp} = 0$),

$$\Delta x = \varepsilon G_c$$

$$= \left(\Delta h_{sp} - \Delta h\right) G_c$$

$$= -\Delta h\, G_c \quad (29)$$

By combining Eqs. (27) and (29), we have

$$\frac{\Delta x}{\Delta q_{in}} = \frac{-G_d G_c}{1 + G_c G_p}$$

$$= \frac{\left(\frac{-K_d \tau_I}{K_p}\right) s + \left(\frac{-K_d}{K_p}\right)}{\tau^2 s^2 + 2\zeta\tau s + 1} \quad (30)$$

If a step change in the disturbance variable with the magnitude **b** (Δq_{in} = **b**/s) is made, the transfer function for Δx becomes

$$\Delta x = \frac{\left(\frac{-K_d \tau_I b}{K_p}\right)}{\tau^2 s^2 + 2\zeta\tau s + 1} + \frac{\left(\frac{-K_d b}{K_p}\right)}{s\left(\tau^2 s^2 + 2\zeta\tau s + 1\right)}$$

$$= \left(\frac{-K_d \tau_I b}{K_p}\right) Y_{imp} + \left(\frac{-K_d b}{K_p}\right) Y_{step} \quad (31)$$

Again, the change in discharge flow rate Δq is simply a linear combination of Δx and Δh as expressed in Eq. (7).

IMC TUNING RULE

If the internal model control (IMC) tuning method is used, the tuning parameters of a PI-controller for a first-order-plus-dead-time process model are

$$K_c = \frac{1}{K_p} \frac{\tau_p}{\left(\theta_p + \tau_f\right)} \quad (32)$$

$$\tau_I = \tau_p \quad (33)$$

where τ_f is the desired closed-loop time constant, an adjustable parameter; and θ_p is the dead time of the process model G_p (θ_p = 0 for a pure first-order model in this work).[4] For aggressive control, a small value of τ_f is chosen. Conversely, a large τ_f value will result in conservative control. By substituting Eqs. (32) (with θ_p = 0) and (33) into Eqs. (19) and (20), we have

$$\tau = \sqrt{\tau_p \tau_f} \quad (34)$$

$$\zeta = \frac{\tau_f + \tau_p}{2\sqrt{\tau_f \tau_p}} \quad (35)$$

SIMULATION

The following operating conditions are chosen for simulation:

- A = cross-sectional area of the open tank = 0.1 m^2
- $q_{in,ss}$ = q_{ss} = steady-state incoming and discharge flow rates = 0.001 m^3 / sec
- k = lumped valve constant = 0.002 $m^{2.5}$ / sec
- x_{ss} = steady-state valve stem position = 0.5

Under these conditions, we have

- h_{ss} = steady-state liquid height = 1.0 m [Eq. (1)]
- K_p = Process gain = - 4 m [Eq. (11)]
- τ_p = 200 sec [Eqs. (10) and (12)]
- K_d = 2.0 × 10^3 sec/m^2 [Eqs. (10) and (15)]

Since most level-control systems in industry deal with regulatory problems, simulation is done here by assuming that a step change of 0.0001 m^3 /sec is made in the incoming flow rate of liquid (*i.e.*, **b** = 0.0001 m^3/sec, or 10% change of the initial steady-state condition) at t = 0. A feedback PI controller is used to reject the disturbance caused by a change in the incoming flow rate, while attempting to maintain the liquid level at the setpoint. Table 1 lists the proportional gain K_c at various values of the desired closed-loop time constant τ_f, as well as the time constant τ and damping factor ζ of the closed-loop second-order characteristic equation with the integral time τ_I set at τ_p (= 200 sec). Using the VBA created by the author for Y_{imp} and Y_{step} for various ranges of ζ values, the responses of liquid level, control valve stem position, and discharge flow rate, all expressed in deviation quantities, are calculated and plotted in Figures 3-5. It is noted that the IMC tuning rule results in a critically damped or overdamped closed-loop characteristic equation with the damping factor $\zeta \geq 1$ (Table 1).

From Figure 4, the smaller the τ_f value (the greater the K_c value), the faster the response of the control valve, as expected. As a result, the liquid level returns to the setpoint value and the discharge flow rate reaches the new steady-state value more quickly (Figures 3 and 5). In other words, a more aggressive control action or tighter control favors both the performance of level control and the response of the discharge flow rate in the disturbance rejection case.

Interested readers may wish to explore the dynamics of

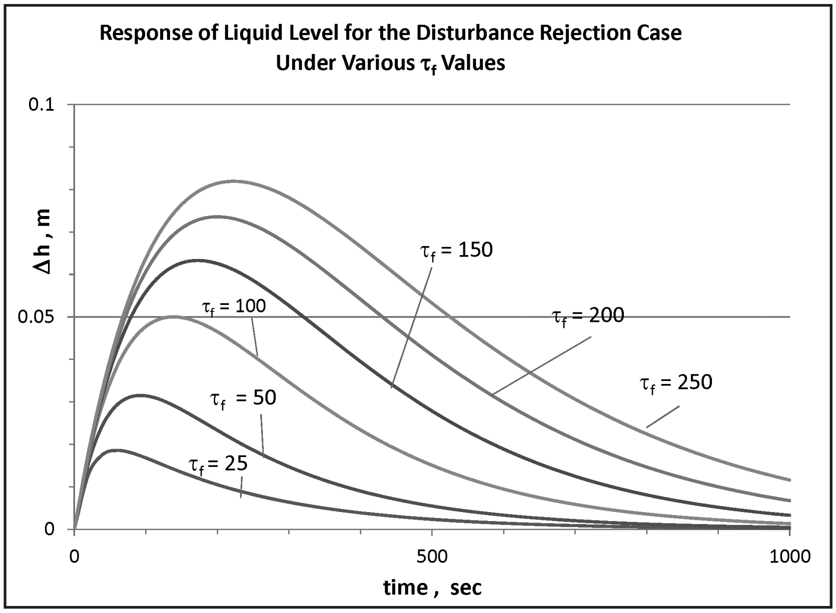


Figure 3.

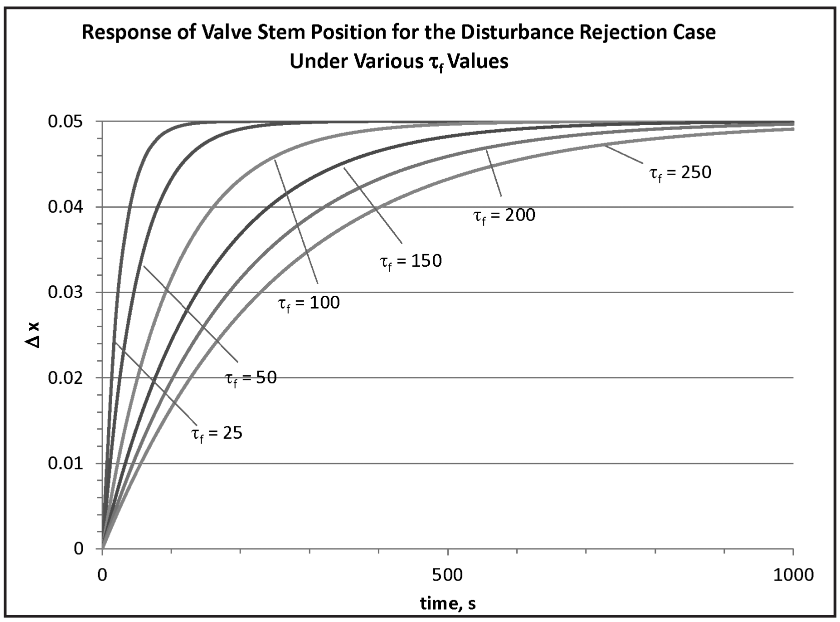


Figure 4.

Interested readers may wish to explore the dynamics of level, control valve stem position, and discharge flow rate in the setpoint-tracking case using the equations developed in this work.

level, control valve stem position, and discharge flow rate in the setpoint-tracking case using the equations developed in this work.

DISCUSSION

If the control valve is located L (m) below the bottom, the models developed in this work can be easily modified with h replaced by (h + L) and h_{ss} by (h_{ss} + L). But the symbol Δh stays the same.

In the author's CHE 460 (Chemical Process Control) class, all three control systems illustrated in Figure 1 are covered. Since very few textbooks cover the details of the system illustrated in Figure 1(c), the author developed this module for students to learn how to derive process and disturbance transfer functions by applying the linearization skill (to tackle the nonlinear term of discharge flow rate through the control valve at the bottom of the tank). The performance of the feedback PI-controller is evaluated not just for the liquid level, but also for the control valve stem position and the discharge flow rate. The IMC control strategy is stressed when selecting control parameters K_c and τ_I.

The first-order differential equation is analogous to that for a simple RC-circuit in which a voltage source is applied to a series combination of a resistance R and a capacitance C. This circuit has a dynamic first-order time constant of RC.[5] Therefore, we may consider the variable R [Eq. (10)] as the linearized resistance and the cross-sectional area of the tank, A, as the capacitance of the level process.

If the expression for time constant τ_p [Eq. (12)] is manipulated in the following manner, it may provide further insight into the nature of the time constant:

TABLE 1

Proportional gain K_c of the PI controller at various values of the desired closed-loop time constant τ_f. By choosing integral time $\tau_I = \tau_p = 200$ s, the resultant second-order time constant τ and damping factor ζ of the closed-loop characteristic function are calculated and used in simulation.

τ_f (sec)	25	50	100	150	200	250
K_c (1/m) (Eq. 32)	-2.00	-1.00	-0.500	-0.333	-0.250	-0.200
τ (sec) (Eq. 34)	70.7	100	141	173	200	223
ζ (Eq. 35)	1.59	1.25	1.06	1.01	1.00	1.01

$$
\begin{aligned}
\tau &= AR \\
&= A\frac{2\sqrt{h_{ss}}}{k\,x_{ss}} \\
&= 2\frac{A\,h_{ss}}{k\,x_{ss}\sqrt{h_{ss}}} \\
&= 2\frac{\text{Steady-state Volume of Liquid in the Tank}}{\text{Steady-state Discharge Flow Rate [Eq. (1)]}} \qquad (36)
\end{aligned}
$$

It is interesting to find that the time constant τ_p is twice as much as the "holding time" (the time needed to drain the liquid from the tank at the steady-state discharge flow rate). Furthermore, the fact that the time constant τ_p, the process gain K_p, and the disturbance gain K_d, all depend on the steady-state or operating condition, further illustrates the nonlinear nature of the dynamic models in this system.

One merit of using the IMC method to tune the PI-controller for a pure first-order system is that the closed-loop dynamics is always critically or over-damped ($\zeta \geq 1$). Therefore, the risk of the final control element becoming saturated due to over-shooting (in the event of under-damped response when large $|Kc|$ and small τ_I are used) may be minimized. As seen in Figure 4, the maximum response of the control valve is $\Delta x = +0.05$ from the initial steady-state condition of $x_{ss} = 0.5$, or x (final) = 0.55. Since the response pattern never overshoots even when tight control (with $\tau_I = \tau_p = 200$ sec and "aggressive" tuning parameter $\tau_f = 25$ sec) is implemented, the valve stem position always stays within the saturation limits (between x = 0 and x = 1). Care must be taken, however, when the unit is operating at the initial valve stem position closer to x = 0 or x = 1 and/or the integral time τ_I is set at smaller values that may lead to an underdamped response.

In the literature, optimization for level control in the pumped tank case [Figure 1(b)] has been explored.[6] The performance of a level-control system does not only consider the response of level alone, but also takes into account the dynamics of the discharge flow rate. Interested workers may develop a similar optimization scheme for the gravity-drained case in Figure 1(c).

CONCLUSIONS

1. The process model and the disturbance model for the level control system with the control valve located at the bottom of the tank can be derived after the governing equation of the discharge flow rate is linearized.
2. Parameters for the first-order process model and disturbance model can be determined from the steady-state condition (or the null operating condition) using the open-loop transfer functions derived in this work.
3. By using the IMC turning rule, integral time τ_I of the feedback PI controller is set at τ_p. The proportional gain K_c can be determined at various values of the desired closed-loop time constant τ_f. The resultant second-order characteristic equation for the closed-loop system is critically or over-damped with $\zeta \geq 1$.
4. With $\tau_I = \tau_p$, aggressive control (small τ_f and large K_c) favors both the level control and the response of discharge flow rate in the disturbance rejection case.

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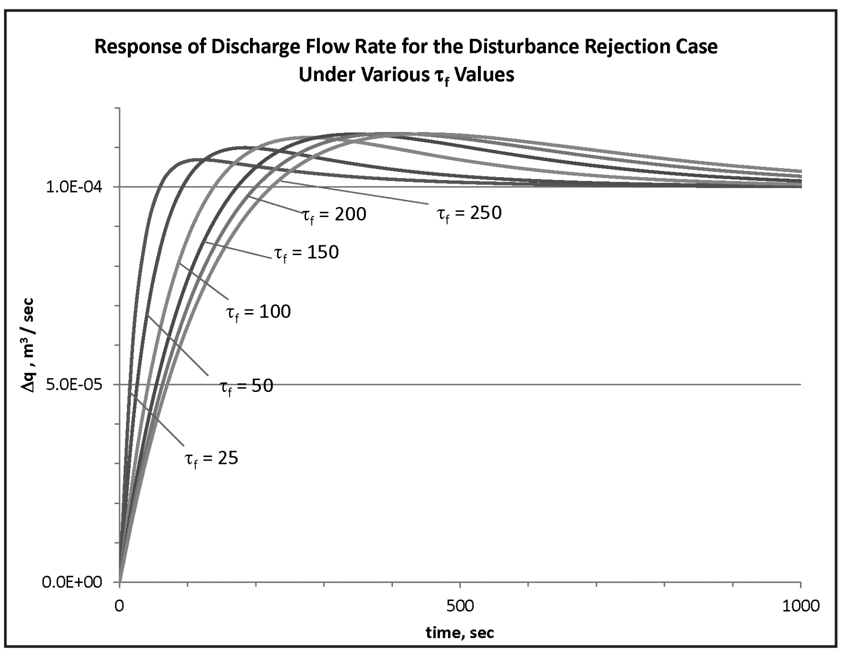

Figure 5.