ChE problems for teachers

PRAIRIE DOG APPENDIX*

R. L. KABEL
Pennsylvania State University
University Park, PA 16802

SOLUTION:

We see that the wind velocity increases as a logarithmic function of the height above the earth's surface. Because of the higher velocity at 2.5 m than at 0.5 m there will be a lower pressure at the top of the tube than at the bottom. This pressure difference will induce an upward flow in the tube.

A Reynolds Number can be calculated for flow through the tube:

$$Re = \frac{D\langle v\rangle\rho}{\mu} = \frac{0.01\text{m}(1\text{ ms}^{-1})(1.2\text{ kg m}^{-3})}{1.8(10^{-5}\text{ kg m}^{-1}\text{ s}^{-1})} = 667$$

which indicates laminar flow.

Thus the Hagen-Poiseuille equation can be used to find the pressure drop. Eq. 2.3-19 Bird, et. al. (or the Prairie Dog problem) gives

$$Q = \frac{\pi\Delta p R^4}{8\mu L} \quad \text{or} \quad \Delta p = \frac{Q8\mu L}{\pi R^4}$$

$$Q = \langle v\rangle A = \langle v\rangle\pi R^2$$
$$= 1\text{ ms}^{-1}(\pi)(0.005^2\text{ m}^2) = 7.85(10^{-5}\text{ m}^3\text{ s}^{-1})$$
$$\mu = 1.8(10^{-5}\text{ kg m}^{-1}\text{ s}^{-1})$$
$$L = 2\text{ m}$$
$$R = 0.005\text{ m}$$

Substituting, we obtain

$$\Delta p = 11.52\text{ kg m}^{-1}\text{ s}^{-1}$$

This pressure drop can be related to horizontal wind velocity by the Bernoulli equation:

$$\Delta\left(\frac{1}{2}\frac{\langle v^3\rangle}{\langle v\rangle} + \hat{\phi}\right) + \int_{p_1}^{p_2}\frac{1}{\rho}dp + \hat{W} + \hat{E}_v = 0$$

Neglecting friction and work, $\hat{E}_v$ and $\hat{W}$ are zero. If the air can be assumed to be incompressible under these conditions, ρ is constant and

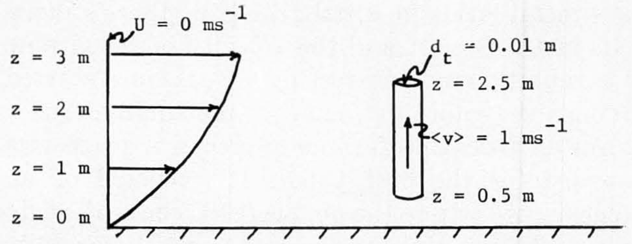

FIGURE 1

$$\int_{p_1}^{p_2}\frac{dp}{\rho} = \frac{p_2 - p_1}{\rho}$$

The potential energy difference $\Delta\hat{\phi}$ between points is negligible and for turbulent flow of air, $\langle v^3\rangle/\langle v\rangle \cong \langle v\rangle^2$, and

$$\langle v_2\rangle^2 - \langle v_1\rangle^2 = \frac{2(p_1 - p_2)}{\rho}$$

Taking point (1) at the bottom and point (2) at the top, both terms are positive. Thus we have one equation and two unknowns. The logarithmic velocity profile

$$\frac{U(z)}{U_*} = \frac{1}{k}\ln\frac{z}{z_0} \quad \text{or} \quad U(z) = \frac{U_*}{0.4}\ln\left[\frac{z}{0.04}\right]$$

provides two more equations but only one more unknown, U_*. Since U in the log velocity profile and $\langle v\rangle$ in the Bernoulli equation are the same thing (i.e. the horizontal wind velocity) we can combine these equations.

$$\langle v_2\rangle = \frac{U_*}{0.4}\ln\left[\frac{2.5}{0.04}\right] = 10.34\,U_*$$

$$\langle v_1\rangle = \frac{U_*}{0.4}\ln\left[\frac{0.5}{0.04}\right] = 6.31\,U_*$$

$$(10.34\,U_*)^2 - (6.31\,U_*)^2 = \frac{2(p_1 - p_2)}{\rho}$$

$$67.10\,U_*^2 = \frac{2(11.52\text{ kg m}^{-1}\text{ s}^{-2})}{1.2\text{ kg m}^{-3}}$$
$$= 19.20\text{ m}^2\text{ s}^{-2}$$

$$U_* = \sqrt{\frac{19.20}{67.10}} = 0.535\text{ ms}^{-1}$$

This value of U_* can now be used in the velocity profile to get velocity at 3 m.

$$U(z) = \frac{U_*}{k}\ln\frac{z}{0.04} = \frac{0.535}{0.4}\ln\frac{3}{0.04}$$
$$= 5.77\text{ ms}^{-1}$$

□

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