AN ALTERNATE METHOD FOR TEACHING AND IMPLEMENTING DIMENSIONAL ANALYSIS

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When my daughter was a little girl, her mother asked, "Whom do you love most, mommy or daddy?" My daughter gave the politically correct answer "I love you both equally." But then her mother (who is a mathematician!) asked, "But who is more equal?" My daughter tactfully responded, "You are, mommy!" Some might contend that dimensional analysis is elementary and that all approaches are equal. If so, my response is that there is a "more equal" approach if one can write down the describing equations. This approach is an alternative to the Pi Theorem method that involves the following three steps

1. *List all qualities on which the phenomenon depends.*
2. *Write the dimensional formula for each quantity.*
3. *Demand that these quantities be combined into a functional relation that remains true independently of the size of the units.*

In Step 3, one invokes the Pi Theorem, which states that n - m dimensionless groups are formed from n quantities expressed in terms of m units. A proof of the Pi Theorem and discussion of the special case n = m are given in Bridgman.[1]

Unfortunately, using the Pi Theorem approach is not always straightforward. For example, how do we select the quantities? When do we include dimensional constants such as g_c (Newton's Law constant), R (gas constant), etc? How are dimensionless quantities such as angles involved? How many units must be considered? For example, force can be considered a primary quantity expressed in units of its own kind (*e.g.,* Newton's), or a secondary quantity expressed in terms of mass, length, and time (*e.g.,* kg·m/s^2). This problem also arises with quantities involving energy or temperature units, since both can be considered either as primary or secondary quantities. If one can write the describing equations, the approach proffered here can be used to avoid the aforementioned difficulties.

AN ALTERNATIVE METHOD FOR DIMENSIONAL ANALYSIS

The procedure for this method is as follows:

1. Write the algebraic and/or differential equations needed to solve the problem.
2. Write any required initial, boundary, and auxiliary conditions.
3. Use all available information in order to simplify the equations in steps 1 and 2.
4. Define dimensionless dependent and independent variables using arbitrary scale and reference factors.
5. Nondimensionalize the equations, initial, boundary, and auxiliary conditions in steps 1 and 2.
6. Determine the scale and reference factors by setting dimensionless groups equal to one (for scale factors) or zero (for reference factors); this yields the minimum parametric representation in the form

$$f(\Pi_1,\Pi_2,\ldots,\Pi_k)=0 \quad (1)$$

   where Π_i denotes a dimensionless group. These Π_i s include dimensionless groups formed from combinations of the physical and geometric quantities and any dimensionless independent variables; the latter will not appear if they are integrated out or evaluated at fixed spatial or temporal conditions.

7. The dimensionless groups in step 6 are not unique; it

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may be advantageous to isolate two dimensional quantities into one group (if possible) in order to determine their interdependence. This is done by forming a new group from the k dimensionless groups via the operation

$$\Pi_p = \beta \bullet \Pi_1^a, \Pi_2^b, \ldots, \Pi_k^j \tag{2}$$

where β is a constant, and a,b,...,j are constants chosen to isolate the desired quantities into the new dimensionless group Π_p; one can use this new group along with any k - 1 of the original groups; however, this operation cannot result in eliminating a dimensional quantity from the analysis.

8. The number of groups can be reduced further when a Π_i is either very large or very small by expanding Eq. (1) in a Taylor series in the small (or reciprocal of a large)

$$f(\Pi_1, \Pi_2, \ldots, \Pi_k) = f\Big|_{\Pi_i=0} + \frac{\partial f}{\partial \Pi_i}\Big|_{\Pi_i=0} \Pi_i + O(\Pi_i^2) \tag{3}$$

If Eq. (3) can be truncated at the first term, the correlation will be a function of k - 1 Π_i s.

This method is closely related to scaling analysis.[2,3] It differs, however, in that no attempt is made to insure that the dimensionless quantities are of order one.

This approach is not new—indeed, Bird, *et al.*, have outlined the technique.[4] Hellums and Churchill[5] also used it to achieve the minimum parametric representation and to identify similarity transformations. This approach has also been suggested in two articles in *Chemical Engineering Education.* Andrews[6] commented, "The subject is best taught by writing down known equations as relations between dimensionless groups," and Churchill[7] stated "Dimensional analysis is most powerful when it is applied to a complete mathematical model in algebraic (differential and/or integral) form..."

Despite this recognition that dimensional analysis is "best taught" and "most powerful" when applied to a complete model, no article has appeared in *Chemical Engineering Education* describing the approach in detail. The latter is the principal goal of this paper. It will also indicate how this approach, when combined with asymptotic analysis, can lead to useful limiting forms. In addition, this paper will indicate how dimensional analysis can be combined with empirical results to obtain information about the functional form of a dimensionless correlation. Finally, it will dispel the notion that dimensional analysis is somehow limited to fluid dynamics and will provide examples drawn from heat and mass transfer!

This alternative method will be illustrated via four examples. The first will be shown in detail while the others will be outlined. The Pi Theorem approach will also be applied to each example, but in doing this, a less general approach typical of that often used by students who have limited experience will be invoked. The author has found that the Pi Theorem can be used to obtain the same result as that of the alternative method proffered here. Far more physical insight is required, however, to achieve the most general result using the Pi Theorem. There are two principal problems in using the Pi Theorem. The first involves Step 3 and the second relates to choosing the proper units. Some believe that the choice of fundamental units is arbitrary. This misconception is the source of much confusion concerning the Pi Theorem and is the cause of many of its alleged violations. For example, in the system of statics, one **must** use the units of force, mass, length, and time, and **should not** introduce the dimensional constant g_c. In contrast, in the system of dynamics, if one introduces force, mass, length, and time as the units, one **must** introduce the dimensional constant g_c, since these units are interrelated by Newton's law of motion. For example, one must introduce g_c if one uses SI units for the quantities involved in a dynamical system, since this system considers force, mass, length, and time as primary quantities.

These subtle aspects of dimensional analysis are discussed in Bridgman,[1] which should be required reading for anyone interested in dimensional analysis. But the need to know these subtleties can be avoided by using the alternative approach suggested here. As such, this alternative method is ideally suited as a teaching tool to give students a working knowledge of dimensional analysis, Its implementation, as well as other judicious operations useful in dimensional analysis, will be illustrated.

TERMINAL VELOCITY OF A SPHERICAL PARTICLE FALLING THROUGH A VISCOUS LIQUID

We seek to correlate the terminal velocity, V_t, of a spherical particle of radius R and density ρ_s falling owing to gravitational acceleration, g, through an incompressible Newtonian liquid having density ρ and viscosity μ, as shown in Figure 1. We obtain V_t from a force balance on the sphere given by

$$-\iint_S \vec{\delta}_z \cdot \vec{\delta}_r \left[P\underline{\underline{\delta}} - \mu\left(\nabla\vec{V} + \nabla\vec{V}^* \right) \right]_{r=R} dS + (\rho_s - \rho) g \frac{4}{3}\pi R^3 = 0 \quad (4)$$

where $\vec{\delta}_i$ is the unit vector in the i-direction, S is the surface area, $\underline{\underline{\delta}}$ is the identity tensor, P is the dynamic pressure, $\vec{V}$ is the fluid velocity, and * denotes the transpose. In order to carry out the integration in Eq. (4), one would have to solve the equations of motion in spherical coordinates with boundary conditions consisting of no-slip at the sphere surface and a far-field velocity condition. In a coordinate system attached to the sphere, these are given by

$$\rho\vec{V}\cdot\nabla\vec{V} = -\nabla P + \mu\nabla^2\vec{V} \quad (5)$$

$$\vec{V} = 0 \quad \text{at} \quad r = R \quad (6)$$

$$\left.\begin{aligned} \vec{V}\cdot\vec{\delta}_r &= -V_t \cos\theta \\ \vec{V}\cdot\vec{\delta}_\theta &= -V_t \sin\theta \end{aligned}\right\} \text{ as } r \to \infty \quad (7)$$

where r and θ denote the radial and circumferential coordinates, respectively.

Define the following dimensionless variables:

$$\hat{\vec{V}} \equiv \frac{\vec{V}}{V_s} \qquad \hat{P} \equiv \frac{P}{P_s} \qquad \hat{\nabla} \equiv L_s\nabla \qquad \hat{S} \equiv \frac{S}{L_s^2} \quad (8)$$

where ˆ denotes a dimensionless variable, and V_s, P_s, and L_s denote velocity, pressure, and length scales that will be chosen to obtain the minimum parametric representation. Introducing these into Eqs. (4-7) and dividing through by the dimensional coefficient of one term in each equation yields

$$-\iint_{\hat{S}} \vec{\delta}_z \cdot \vec{\delta}_n \left[\frac{P_s L_s}{\mu_s V_s}\hat{P}\underline{\underline{\delta}} - \left(\hat{\nabla}\hat{\vec{V}} + \hat{\nabla}\hat{\vec{V}}^* \right) \right]_{\hat{r}=\frac{R}{L_s}} d\hat{S} + \frac{(\rho_s - \rho) g \frac{4}{3}\pi R^3}{\mu V_s L_s} = 0 \quad (9)$$

$$\frac{\rho V_s L_s}{\mu}\hat{\vec{V}}\cdot\hat{\nabla}\hat{\vec{V}} = -\frac{P_s L_s}{\mu V_s}\hat{\nabla}\hat{P} + \hat{\nabla}^2\hat{\vec{V}} \quad (10)$$

$$\hat{\vec{V}} = 0 \quad \text{at} \quad \hat{r} = \frac{R}{L_s} \quad (11)$$

$$\left.\begin{aligned} \hat{\vec{V}}\cdot\vec{\delta}_r &= -\frac{V_t}{V_s}\cos\theta \\ \hat{\vec{V}}\cdot\vec{\delta}_\theta &= -\frac{V_t}{V_s}\sin\theta \end{aligned}\right\} \text{ as } \hat{r} \to \infty \quad (12)$$

One possible set of scale factors is obtained by setting the following dimensionless groups equal to one:

$$\frac{R}{L_s} = 1 \Rightarrow L_s = R \qquad \frac{V_t}{V_s} = 1 \Rightarrow V_s = V_t \qquad \frac{P_s L_s}{\mu V_s} = 1 \Rightarrow P_s = \frac{\mu V_t}{R} \quad (13)$$

Hence, the solution to Eq. (10) will depend on $\hat{r}$, θ, and the dimensionless group $\rho V_t R/\mu$. When this solution is substituted into Eq. (9), evaluated at $\hat{r} = 1$, and integrated over the surface area, the resulting solution for the dimensionless terminal velocity can be correlated in terms of the following two dimensionless groups:

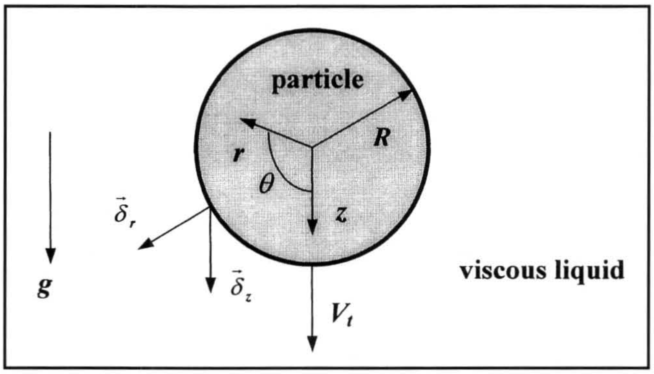

Figure 1.
Spherical particle falling through a viscous liquid.

$$\Pi_1 \equiv \frac{(\rho_s - \rho) g R^2}{\mu V_t} \quad \text{and} \quad \Pi_2 \equiv \frac{\rho V_t R}{\mu} \text{(a Reynolds number)} \quad (14)$$

Hence, either data or a numerical solution for V_t can be correlated in terms of Π_1 and Π_2. These are not optimal, however, if one is seeking a correlation for V_t, since it appears in both groups. By invoking the transformation in Step 7 with a=1 and b=1 in Eq. (2), a new dimensionless group, Π_3, not containing V_t, can be obtained:

$$\Pi_3 \equiv \frac{(\rho_s - \rho)\rho g R^3}{\mu^2} \quad (15)$$

Hence, data or numerical results for V_t can be correlated in terms of Π_3 and either Π_1 or Π_2.

A naïve application of the Pi Theorem with n=6 and m=3 (or n=7 and m=4 if F is also used as a unit and g_c included as a quantity) indicates that the correlation for V_t requires three rather than two dimensionless groups. In order to obtain the most general result from the Pi Theorem, one must recognize that g appears as the product $g(\rho_s - \rho)$. The alternative approach suggested here avoids these subtleties associated with the Pi Theorem method.

Standard references[4] suggest that V_t can be correlated in terms of just Π_1; that is

$$V_t = \frac{2R^2 g(\rho_s - \rho)}{9\mu} \Rightarrow \Pi_1 = \frac{(\rho_s - \rho) g R^2}{\mu V_t} = \frac{9}{2} \quad (16)$$

This is for the special case of creeping flow for which the intertia terms can be neglected, however. Hence, Π_2 (or equivalently Π_3) no longer appears in the minimum parametric representation. Note, a naïve application of the Pi Theorem would suggest three dimensionless groups (n=6 and m=3). But the Pi Theorem will predict one group if one recognizes that g appears as the product $g(\rho_s - \rho)$ and that F **must** be introduced as a unit (without including g_c), since creeping flow is a problem in statics. Clearly, the alternative approach proffered here obviates the need to be aware of these special considerations required to get the most general result using the Pi Theorem.

A simpler way to obtain this result is to use Step 8. Since creeping flow implies $Re = \Pi_2 \to 0$, the expansion must use groups Π_1 and Π_2. In the limit of $\Pi_2 \to 0$ one obtains that V_t can be correlated solely in terms of Π_1.

FALLING-HEAD METHOD FOR DETERMINING THE PERMEABILITY OF A POROUS MEDIUM

The falling-head method is used to determine permeability of soils. This test, shown in Figure 2, involves driving a pipe of radius R into the soil until it penetrates the water table, which is shown here at the exit of the tube. The pipe is filled with water to a height L_i and the time t_D required to drain it to a height L_f is measured. The draining time t_D is related to the axial velocity V_z and Darcy's law via the equation

$$(L_i - L_f) = -\int_0^{t_D} V_z\Big|_{z=0} dt = \int_0^{t_D} \frac{k}{\mu}\left(\frac{\partial P}{\partial z}+\rho g\right)\Bigg|_{z=0} dt \qquad 0 \le r \le R \qquad (17)$$

where k is the permeability, μ is the viscosity, ρ is the density, and g is the gravitational acceleration. The incompressible continuity equation implies that the pressure, P, is obtained from a solution to the axisymmetric Laplace's equation in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)+\frac{\partial^2 P}{\partial z^2}=0 \qquad (18)$$

This is subject to the boundary conditions

$$P = P_{atm} + \rho g L(t) \quad \text{at} \quad z=0, \quad L_f \le L(t) \le L_i \quad 0 \le r \le R \qquad (19)$$

$$P = P_{atm} \quad \text{at} \quad z=0, \quad R \le r < \infty \qquad (20)$$

$$V_z = -\frac{k}{\mu}\frac{\partial P}{\partial z} = 0 \quad \text{as} \quad z \to \infty, \quad 0 \le r < \infty \qquad (21)$$

$$V_r = -\frac{k}{\mu}\frac{\partial P}{\partial r} = 0 \quad \text{at} \quad r=0, \quad -\infty < z \le 0 \qquad (22)$$

$$P = P_{atm} - \rho g z \quad \text{as} \quad r \to \infty, \quad -\infty < z \le 0 \qquad (23)$$

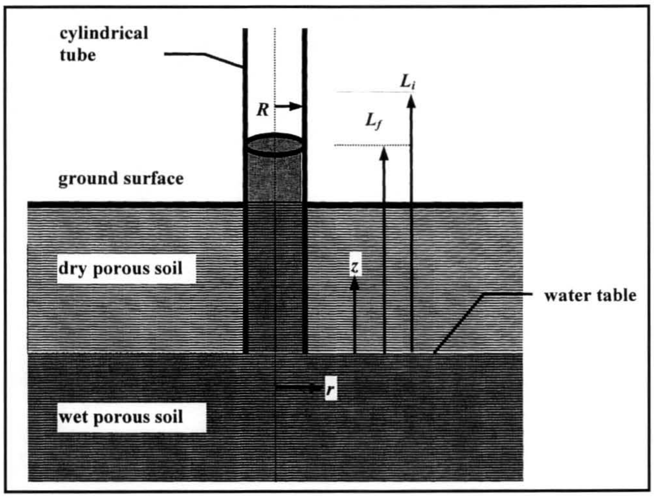

Figure 2. *Falling-head apparatus for measuring the permeability of a soil.*

Applying Steps 4, 5, and 6 leads to the following dimensionless variables:

$$\hat{P} \equiv \frac{P - P_{atm}}{\rho g L_i} \quad \hat{r} \equiv \frac{r}{R} \quad \hat{z} \equiv \frac{z}{L_i} \quad \hat{L} \equiv \frac{L}{L_i} \quad \hat{t} \equiv \frac{k\rho g t}{\mu(L_i - L_f)} \qquad (24)$$

Note that the solution to the dimensionless form of Eq. (18) will give the pressure as a function of $\hat{t}$, $\hat{r}$, and $\hat{z}$. But when the pressure is substituted into the dimensionless form of Eq. (17), evaluated at $\hat{z}$=0, and the integration carried out, the resulting solution will contain only three dimensionless parameters. This is the minimum parametric representation and implies that k is correlated in terms of t_D as follows:

$$\Pi_1 \equiv \frac{k\rho g t_D}{\mu(L_i - L_f)} = f\left(\frac{R}{L_i}, \frac{L_f}{L_i}\right) = f(\Pi_2, \Pi_3) \qquad (25)$$

A naïve application of the Pi Theorem would give five dimensionless groups (*i.e.,* n=8 and m=3)! Equation (25) can be obtained from the Pi Theorem if one recognizes that $k\rho g/\mu$ can be considered as a single quantity, thereby giving two groups (*i.e.,* m=5 and n=2).

Equation (25) can be cast into a more useful form by using an empirical correlation obtained using water and a 5-cm radius pipe for a soil having a k=5.9x10^{-6}cm^2:[8]

$$t_D = 4.94\, \ell n\left(\frac{L_i}{L_f}\right) \qquad (26)$$

It is convenient to recast Eq. (25) in terms of a new dimensionless group that does not contain L_i and L_f:

$$\Pi_4 \equiv \frac{\Pi_1}{\Pi_2} = \frac{t_D k\rho g}{\mu R\left(1 - \frac{L_f}{L_i}\right)} = f(\Pi_2, \Pi_3) \Rightarrow \frac{t_D k\rho g}{\mu R} \equiv \Pi_5 = f(\Pi_2, \Pi_3) \qquad (27)$$

If R<<L_i, then $\Pi_2 << 1$, and it follows that

$$\frac{t_D k\rho g}{\mu R} \equiv \Pi_5 = f(\Pi_3) \qquad (28)$$

Comparing Eq. (28) with Eq. (26) then implies that

$$t_D = \Pi_5\left(\frac{\mu R}{k\rho g}\right) = -4.94\, \ell n\, \Pi_3 \qquad (29)$$

Hence, if $\Pi_2 << 1$, the generalized correlation relating k and t_D is obtained by substituting values for the quantities in Eq. (29) and is given by

$$\Pi_5 = -\frac{4.94\left(5.9\times10^{-6}\,cm^2\right)\left(1\frac{g}{cm^3}\right)\left(980\frac{cm}{s^2}\right)}{\left(0.01\frac{g}{cm\cdot s}\right)(5\,cm)}\ell n\, \Pi_3 = -0.572\, \ell n\, \Pi_3 \qquad (30)$$

In this case, an enlightened approach to dimensional analysis in combination with data for a specific falling-head test gives the functional form of the generalized t_D correlation in terms of the relevant parameters; *i.e.,* Eq. (30) applies for any falling-head test for which $\Pi_2 << 1$, irrespective of the fluid, pipe size, and soil.

DESIGN EQUATION FOR ROASTING TURKEY

Mom and Dad are planning to roast turkey for the entire clan and need to know the cooking time, t_C, for the 28-lb bird shown in Figure 3. The cookbook provides the information shown in Table 1.[9] This is your opportunity to impress them with what you have learned by using dimensional analysis to generalize Table 1. Assume that it takes p geometric parameters to characterize the shape of a turkey and that all turkeys are geometrically similar. Hence, the p-1 dimensionless geometric ratios characterizing turkeys will be the same. One need only include one geometric quantity such as some characteristic body dimension, L, along with the quantities that characterize the heat transfer in the dimensional analysis. Roasting turkey involves a constant oven temperature T_S (325°F[9]). The bird is done when the center of the stuffing reaches a temperature T_0 (165°F[9]). The heat transfer is limited by heat conduction through the bird and stuffing, whose thermal conductivities and diffu-sivities are k_B, k_S, and α_B and α_S, respectively, and are assumed constant for all turkeys. Hence, t_C is obtained from a solution to the three-dimensional unsteady-state conduction equation in the turkey and the stuffing:

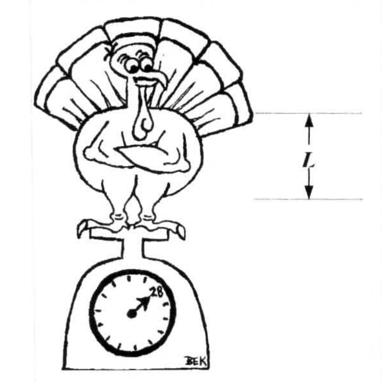

Figure 3. *A very large turkey having a characteristic length L.*

TABLE 1
Timetable for Roasting Turkey[9]

Ready-to-Cook Weight W (lb)	*Approximate Total Cooking Time t_c (hr)*
6 to 8	3 to 3 ½
8 to 12	3 ½ to 4 ½
12 to 16	4 ½ to 5 ½
16 to 20	5 ½ to 6 ½
20 to 24	6 ½ to 7

$$\frac{\partial T}{\partial t} = \alpha_B \nabla^2 T \tag{31}$$

$$\frac{\partial T}{\partial t} = \alpha_S \nabla^2 T \tag{32}$$

The initial and boundary conditions are given by

$$T = T_I \text{ at } t = 0 \tag{33}$$

$$T = T_S \text{ at the surface of the turkey} \tag{34}$$

$$T\Big|_+ = T\Big|_- \text{ at the interface between turkey and stuffing} \tag{35}$$

$$k_B \nabla T\Big|_+ = k_S \nabla T\Big|_- \text{ at the interface between turkey and stuffing} \tag{36}$$

$$\nabla T = 0 \text{ along the plane of symmetry in the turkey} \tag{37}$$

Applying Steps 4, 5, and 6 leads to the following dimensionless variables:

$$\hat{T} \equiv \frac{T - T_I}{T_S - T_I} \quad \hat{t} \equiv \frac{t\alpha_B}{L^2} \quad \hat{\nabla}^2 \equiv L^2 \nabla^2 \tag{38}$$

Introducing these variables into Eqs. (31-37) implies the following minimum parametric representation:

$$\frac{t_C \alpha_B}{L^2} = f\left[\frac{(T_0 - T_I)}{(T_S - T_I)}, \frac{\alpha_B}{\alpha_S}, \frac{k_B}{k_S}, \ldots, \text{geometric ratios}\right] \tag{39}$$

Hence, for geometrically similar turkeys and constant physical properties, $t_C \propto L^2$. For a spherical turkey body, $L \propto W^{1/3}$. Hence, $t_C = KW^{2/3}$, where K is a proportionality constant determined from the data in Table 1. The following correlation fits these data with an $R^2 = 0.994$:

$$t_C = 0.864\, W^{2/3} \tag{40}$$

Hence, Mom and Dad's 28-lb bird will require a t_C = 8 hours. Note that a naïve application of the Pi Theorem would imply five dimensionless groups (*i.e.,* n=9 and m=4) in addition to the geometric ratios.

DESIGN OF A NOVEL MEMBRANE BLOOD OXYGENATOR

A recent patent describes a hollow fiber membrane blood oxygenator that offers a 300% increase in O_2 and CO_2 mass transfer.[10] This is achieved by oscillating the hollow fiber membrane module relative to the blood flow. This enhances the mass transfer on the blood side where O_2 diffusion is limiting. We seek to correlate the mass-transfer coefficient for this device by considering a single oscillating hollow fiber as shown in Figure 4.

An analytical solution has been developed for the hydrodynamics when the membrane tube bundle is pulsated harmonically at a frequency ω and amplitude A; the velocity profile, $\hat{V}_z$, is of the form[11]

$$\hat{V}_z \equiv \frac{V_z}{\overline{V}} = f\left(\frac{r}{R}, \omega t, \frac{\nu}{\omega R^2}, \frac{A\omega}{\overline{V}}\right) \tag{41}$$

where $\overline{V}$ is the volume-average velocity, R is the hollow-fiber radius, ν is the kinematic viscosity of the Newtonian fluid, and t is the time. We seek a correlation for the mass-transfer coefficient defined in terms of the log-mean driving force, ΔC_{lm}, and the overall length of the tube, L, as follows:

$$k_L \equiv \frac{\overline{N}_W}{\Delta C_{lm}} = \frac{\omega D}{2\pi L \Delta C_{lm}} \int_0^{2\pi/\omega} \int_0^L \frac{\partial C}{\partial r}\Bigg|_{r=R} dz\,dt \tag{42}$$

where

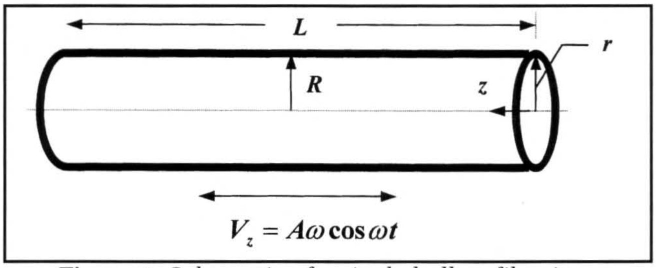

Figure 4. *Schematic of a single hollow fiber in a membrane lung oxygenator.*

$$\Delta C_{lm} \equiv \left[(C_W - C_L) - (C_W - C_O)\right] \Big/ \ell n \frac{(C_W - C_L)}{(C_W - C_O)} \qquad (43)$$

in which $\overline{N}_w$ and C_w are the mass-transfer flux and concentration, respectively, at the blood side of the membrane, C_o and C_L are the concentrations at z=0 and z=L, respectively, where L is the length of the hollow fiber, and D is the binary diffusion coefficient. The concentration in Eq. (42) is obtained from a solution to the axisymmetric form of the advective diffusion equation in cylindrical coordinates:

$$\frac{\partial C}{\partial t} + V_z \frac{\partial C}{\partial z} = \frac{D}{r}\frac{\partial}{\partial r}\left(r \frac{\partial C}{\partial r}\right) \qquad (44)$$

Axial diffusion is neglected based on scaling arguments.[2,3] The boundary and periodic solution conditions are

$$C = C_W \quad \text{at } r = R \qquad (45)$$

$$\frac{\partial C}{\partial r} = 0 \quad \text{at } r = 0 \qquad (46)$$

$$C = C_O \quad \text{at } z = 0 \qquad (47)$$

$$C_{t,r,z} = C\big|_{t+2\pi/\omega,r,z} \qquad (48)$$

Applying Steps 4,5, and 6 leads to the following dimensionless variables:

$$\hat{C} \equiv \frac{C - C_O}{C_W - C_O} \quad \hat{r} \equiv \frac{r}{R} \quad \hat{z} \equiv \frac{z}{L} \quad \hat{t} \equiv \omega t \qquad (49)$$

Introducing these into Eqs. (42) and (44) leads to

$$\frac{k_L R}{D} \equiv Sh = -\frac{\ell n\left(1 - \hat{C}_L\right)}{2\pi \hat{C}_L} \int_0^{2\pi}\int_0^1 \left.\frac{\partial \hat{C}}{\partial \hat{r}}\right|_{\hat{r}=1} d\hat{z}\, d\hat{t} \qquad (50)$$

$$\frac{\omega R^2}{D}\frac{\partial \hat{C}}{\partial \hat{t}} + \frac{2Gz}{\pi}\hat{V}_z \frac{\partial \hat{C}}{\partial \hat{z}} = \frac{\partial}{\partial \hat{r}}\left(\hat{r}\frac{\partial \hat{C}}{\partial \hat{r}}\right) \qquad (51)$$

where $Gz \equiv \pi \overline{V} R^2 / 2DL = \pi R Pe / 2L$ is the Graetz number and Pe is the Peclet number. Equation (50) implies that Sh, the Sherwood number, is a function of the dimensionless groups involved in determining $\hat{C}_L$ and $\partial \hat{C} / \partial \hat{r}$ at the fiber wall and hence will be functions of only the dimensionless groups involved in solving Eq. (51); therefore

$$Sh = f\left(Gz, \frac{\omega R^2}{D}, \frac{\omega R^2}{\nu}, \frac{A\omega}{\overline{V}}\right) \quad \text{or} \quad Sh = f\left(Gz, Sc, \frac{\omega R^2}{D}, \frac{A\omega}{\overline{V}}\right) \qquad (52)$$

where $Sc \equiv \nu / D$ is the Schmidt number, introduced by using the transformation given by Eq. (2). For large Sc (*i.e.,* for blood) we can use the expansion (in Sc^{-1}) suggested by Eq. (3) to conclude that the oxygenator performance can be correlated in terms of only four dimensionless groups. Note that a naïve application of the Pi Theorem would imply that eight dimensionless groups would be required (*i.e.,* n=11 and m=3).

EPILOGUE

The Pi Theorem will yield the minimum number of dimensionless groups, if one can determine the proper quantities and units to use. But this requires considerable physical insight, often well beyond the experience of many students. The alternative method proposed here directly yields the minimum parametric representation without having to invoke any sophisticated reasoning concerning how variables appear in certain combinations or the proper units for the particular physical system. Hence, in the words of my young daughter, it is "more equal!"

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