The object of this column is to enhance our readers' collections of interesting and novel problems in chemical engineering. Problems of the type that can be used to motivate the student by presenting a particular principle in class, or in a new light, or that can be assigned as a novel home problem, are requested, as well as those that are more traditional in nature and that elucidate difficult concepts. Manuscripts should not exceed ten double-spaced pages if possible and should be accompanied by the originals of any figures or photographs. Please submit them to Professor James O. Wilkes (e-mail: wilkes@umich.edu), Chemical Engineering Department, University of Michigan, Ann Arbor, MI 48109-2136.

# USE OF SEQUENTIAL AND NON-DISCIPLINARY PROBLEMS TO TEACH PROCESS DYNAMICS

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This paper illustrates two useful pedagogical techniques for motivating and teaching students that can be easily applied to teaching process dynamics. The two basic ideas are: 1) use situations that are not chemical engineering and 2) use different versions of the same problem sequentially throughout the duration of the course.

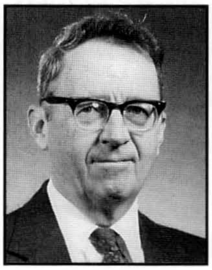
The first helps to motivate students because they can see that the basic principles of developing dynamic mathematical models have wide application in many aspects of life. The second provides the "creative redundancy" that is needed to really understand a subject.

One example of this approach is presented here. There are four similar problems that have slightly different mathematical models and/or boundary conditions:

- The 1805 Battle of Trafalgar (Version 1)
- The Battle of Trafalgar (Version 2)
- The Battle of the North Atlantic (1940)
- The 2200 battle between the Federation fleet of starships, led by Captain Kirk, and the evil Klingon fleet

The originator in chemical engineering of the idea of motivating students by using non-chemical engineer-ing examples was Octave Levenspiel. In his pioneering textbook, *Chemical Reaction Engineering,*[1] he presented a number of problems that were outside the chemical engineering field. As a graduate student studying this book, I found these problems very refreshing. The typical textbook back in those days (and still true for many books today) was dry as dust. The language was very stiff and formal. The use of the first person was unheard of, as was any attempt to inject humor. All the material was straight-line chemical engineering.

Levenspiel changed all that and produced a very "user-friendly" book. His "reactor design" problems included the Battle of Trafalgar, Snake-Eyes Magoo betting habits, investigation of the missing operator by Sherlock Holmes and Dr. Watson, etc. These problems were a great help in letting students understand that the basic principles could be applied to a wide spectrum of life situations. In my own writing over the



***William L. Luyben*** *earned degrees in chemical engineering from Penn State (BS, 1955) and Delaware (PhD, 1963). His industrial experience includes four years with Exxon, four years with DuPont, and three decades of consulting with chemical and petroleum companies. He has taught at Lehigh University since 1967 and has participated in the development of several innovative undergraduate courses, from the introductory course in mass and energy balances through the capstone senior design course and an interdisciplinary controls laboratory.*

last forty years, I have tried to follow Levenspiel's example by using problems drawn from such diverse areas as farming, whiskey making, mechanical and aerospace systems, etc.

In the following sections, I will show how the "Battle of Trafalgar" problem can be extended to teach students the principles of dynamic mathematical modeling and the use of Laplace transforms.

## SEQUENTIAL PROBLEMS

The second idea has grown out of over three decades of undergraduate teaching. The approach is to assign a homework problem and go over its solution in class. Then in the first examination give a similar problem that is a slight extension of the first, and in the second examination given another similar problem that adds different features to the mathematical model that must be derived and solved. By the time the students get to the end of the course, they have figured out that there will be a similar problem on the final examination, and they all know how to solve it.

The principle behind this approach is "creative redundancy." How many times have you heard the remark "I really didn't understand thermodynamics until I took the third thermo course." Repetition is a fundamental approach to learning. The idea is to make sure it is not boring.

This paper presents one example of this sequential, non-disciplinary problem approach. The problems are published in Luyben and Luyben.[2]

## HOMEWORK PROBLEM

This problem is assigned early in the course after the fundamentals of dynamic modeling have been reviewed and illustrated with several chemical engineering processes.

### ■ *Problem Statement*

Solve the following problem, which is part of a problem given in Levenspiel's *Chemical Reaction Engineering,* using Laplace transform techniques. Find an analytical expression for the number of Nelson's ships, $N_{(t)}$ and the number of Villeneuve's ships, $V_{(t)}$ as functions of time.

> *The great naval battle, to be known to history as the Battle of Trafalgar (1805), was soon to be joined. Admiral Villeneuve proudly surveyed his powerful fleet of 33 ships stately sailing in single file in the light breeze. The British fleet under Lord Nelson was now in sight, 27 ships strong. Estimating that it would still be two hours before the battle, Villeneuve popped open another bottle of burgundy and point-by-point reviewed his carefully thought-out battle strategy. As was the custom of naval battles at that time, the two fleets would sail in single file parallel to each other and in the same direction, firing their cannons madly. Now, by long experience in battles of this kind, it was a well-known fact that the rate of destruction of a fleet was proportional to the firepower of the opposing fleet. Considering his ships to be on a par, one-for-one, with the British, Villeneuve was confident of victory. Looking at his sundial, Villeneuve sighed and cursed the light wind; he'd never get it over with in time for his favorite television western. "Oh, well," he sighed, "C'est la vie." He could see the headlines next morning—"British Fleet annihilated, Villeneuve's losses are...." Villeneuve stopped short. How many ships would he lose? He called over his chief bottle-cork popper, Monsieur Dubois, and asked this question. What answer did he get?*

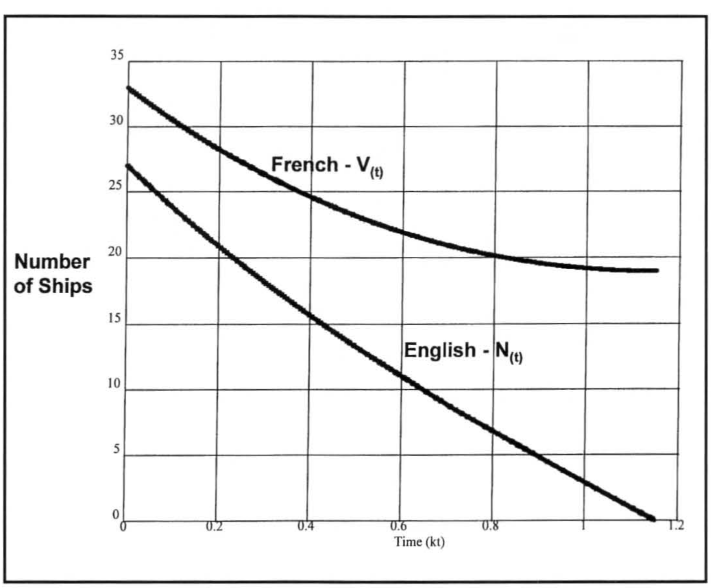



**Figure 1.** *Battle of Tafalgar No. 1.*

### ■ *Problem Solution*

The mathematical model of this problem consists of two linear ordinary differential equations:

$$\frac{dN}{dt} = -kV \tag{1}$$

$$\frac{dV}{dt} = -kN \tag{2}$$

with the initial conditions $N_{(t=0)} = 27$ and $V_{(t=0)} = 33$, and k is a constant equal to the rate of destruction per ship. This rate constant is analogous to the rate constant in a chemical reaction. Note that the actual variables in this problem are discrete (integers), but we are approximating the system with continuous variables to keep the mathematics simple. For reasonably large values (>10), this approximation is probably fairly accurate.

Laplace transforming gives the two algebraic equations

$$sN_{(s)} - 27 = -kV_{(s)} \quad (3)$$

$$sV_{(s)} - 33 = -kN_{(s)} \quad (4)$$

where s is the Laplace transform variable. Combining these equations gives expressions for $N_{(s)}$ and $V_{(s)}$, which can be inverted back into the time domain to obtain $N_{(t)}$ and $V_{(t)}$.

$$N_{(s)} = \frac{27s - 33k}{s^2 - k^2} \quad (5)$$

$$V_{(s)} = \frac{33s - 27k}{s^2 - k^2} \quad (6)$$

$$N_{(t)} = 30e^{-kt} - 3e^{kt} \quad (7)$$

$$V_{(t)} = 3e^{kt} + 30e^{-kt} \quad (8)$$

Note that there are positive eigenvalues. This does not mean that variables become infinite because the solution is limited to finite values of time. The battle ends at $t_F$ when the number of Nelson's ships goes to zero, $N_{(t_F)} = 0$,

$$N_{(t_F)} = 0 = 30e^{-kt_F} - 3e^{kt_F} \quad (9)$$

Solving for $t_F$ gives

$$t_F = \frac{\ell n 10}{2k} \quad (10)$$

The number of Villeneuve's ships left is

$$V_{(t_F)} = 3e^{k(\ell n 10)/2k} - 30e^{-k(\ell n 10)/2k} = 18.95 \quad (11)$$

Therefore, Villeneuve has lost 33-19 = 14 ships. Figure 1 shows the dynamic changes in the number of ships in each fleet.

This problem is assigned and its solution is discussed carefully in class before the first examination.

## EXAMINATION 1 PROBLEM

Now we modify the problem by breaking it into two different battles. The model is the same, but there are two time periods with different initial conditions. Note that some of the characters are personalized to increase the interest level of the class (Bethany Steadman was the student who asked to review the original homework problem).

### ■ Problem Statement

*While Adminral Villeneuve was doing his calculations about the outcome of the Battle of Trafalgar, Admiral Nelson was also doing some thinking. His fleet was outnumbered 33 to 27, so it didn't take a rocket scientist to predict the outcome of the battle if the normal battle plan was followed (the opposing fleets sailing parallel to each other). So Admiral Nelson turned for help to his trusty young Lt. Steadman, who fortunately was an innovative Lehigh graduate in chemical engineering (class of 1796). Steadman opened up her textbook on Laplace transforms and did some back-of-the-envelope calculations to evaluate alternative battle strategies.*

*After several minutes of brainstorming and calculations (she had her PC on board, so she could use MATLAB to aid in the numerical calculations), Lt. Steadman devised the following plan: The British fleet would split the French fleet, taking on 17 ships first and then attacking the other 16 French ships with the remaining British ships. Admiral Nelson approved the plan, and the battle began.*

*Solve quantitatively for the dynamic changes in the number of British and French ships as functions of time during the battle. Assume the rate of destruction of a fleet is proportional to the firepower of the opposing fleet and that the ships are on a par with each other in firepower.*

### ■ Problem Solution

The ordinary differential equations are exactly the same as in the homework problem, but the initial conditions are different. Generalizing the solution for arbitrary initial numbers of ships in each fleet, let $N_{(t=0)} = N_o$ and $V_{(t=0)} = V_o$. The solution is

$$N_{(t)} = \left(\frac{V_o + N_o}{2}\right)e^{-kt} - \left(\frac{V_o - N_o}{2}\right)e^{kt} \quad (12)$$

$$V_{(t)} = \left(\frac{V_o + N_o}{2}\right)e^{-kt} + \left(\frac{V_o - N_o}{2}\right)e^{kt} \quad (13)$$

During the first battle when Nelson takes on half the French fleet, the initial conditions are $N_{(t=0)} = 27$ and $V_{(t=0)} = 17$. The end of this initial battle occurs at $t_{F1}$ when $V_{1(t)} = 0$, or

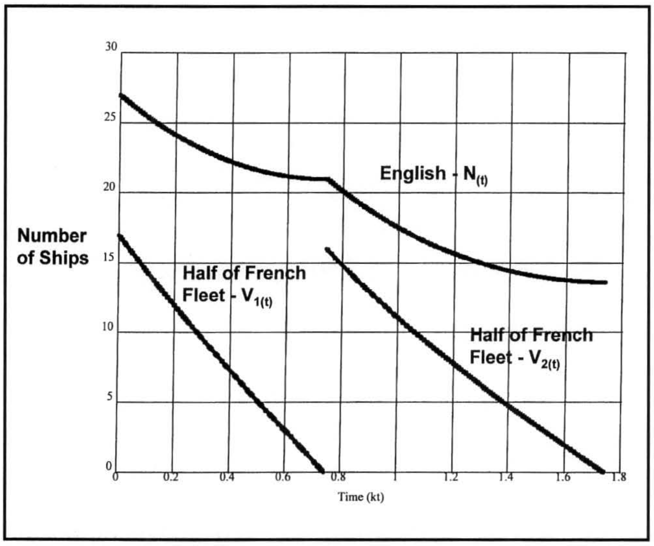



**Figure 2.** *Battle of Tafalgar No. 2.*

$$V_{1(t)} = 0 = \left(\frac{44}{2}\right)e^{-kt_{F1}} + \left(\frac{-10}{2}\right)e^{kt_{F1}} \tag{14}$$

$$t_{F1} = \frac{\ell n(4.4)}{2k} = 0.7408 / k \tag{15}$$

and the remaining number of British ships is

$$N_{(t_{F1})} = \left(\frac{44}{2}\right)e^{-0.7408} + \left(\frac{10}{2}\right)e^{0.7408} = 21 \tag{16}$$

Now the second phase of the battle begins with the initial conditions $N_{(t=0)} = 21$ and $V_{(t=0)} = 16$. The dynamic changes in the number of French ships is

$$V_{(t)} = \left(\frac{37}{2}\right)e^{-kt} + \left(\frac{-5}{2}\right)e^{kt} \tag{17}$$

The time it takes Nelson to completely demolish the French fleet is

$$t_{F2} = \frac{\ell n(37/5)}{2k} = \frac{1.0007}{k} \tag{18}$$

The remaining number of British ships is

$$N_{(t_{F1})} = \left(\frac{37}{2}\right)e^{-1.0007} + \left(\frac{5}{2}\right)e^{1.0007} = 13.5 \tag{19}$$

Figure 2 shows the dynamic changes in the number of ships in each fleet during the two phases of the battle.

So Nelson and the British fleet win the day (with a little help from a Lehigh chemical engineer)!

## EXAMINATION 2 PROBLEM

On the next test the problem is modified to include generation terms in the differential equations in addition to the depletion terms.

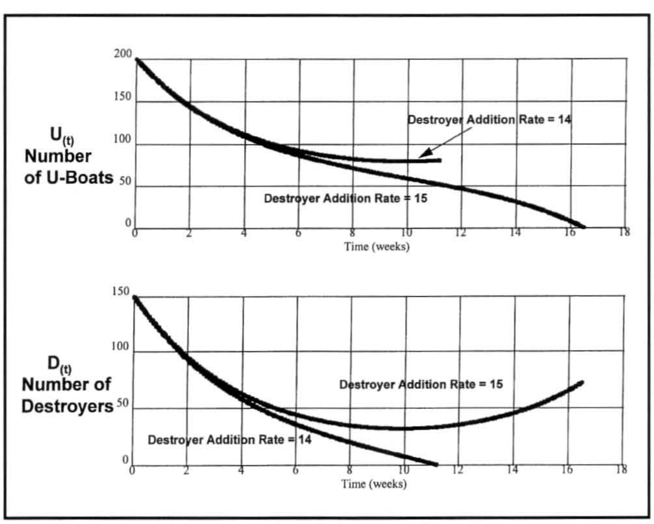



***Figure 3.*** *Battle of the North Atlantic.*

### ■ *Problem Statement*

*The 1940 Battle of the North Atlantic is about to begin. The German submarine fleet consists of 200 U-boats at the beginning of the battle. The British destroyer fleet, under the command of Admiral Steadman (a direct descendant of the intelligence officer responsible for the British victory at the Battle of Trafalgar), consists of 150 ships at the beginning of the battle. The rate of destruction of submarines by destroyers is equal to the rate of destruction of destroyers by submarines: 0.25 ships/week/ship.*

*Germany is launching two new submarines per week and adding them to its fleet. President Roosevelt is trying to decide how many new destroyers per week must be sent to the British fleet under the Lend-Lease Program in order to win the battle. Admiral Steadman claims she needs 15 ships added to her fleet per week to defeat the U-boat fleet. The Secretary of the Navy, William Gustus, claims she only needs 5 ships per week. Who is correct?*

### ■ *Problem Solution*

The dynamic mathematical model describing the number of destroyers, $D_{(t)}$, and the number of U-boats, $U_{(t)}$, is

$$\frac{dD}{dt} = -kU + P_D \tag{20}$$

$$\frac{dU}{dt} = -kD + P_U \tag{21}$$

where k is the rate of destruction (0.25 ships destroyed per week for each ship in the opposing fleet), $P_D$ and $P_U$ are the weekly rate of addition of destroyers and U-boats to the fleets, and time, t, is in weeks. The initial conditions are $D_{(t=0)} = D_o = 150$ and $U_{(t=0)} = U_o = 200$. Laplace transforming and combining gives

$$U_{(s)} = \frac{U_o s^2 + (P_U - kD_o)s - kP_D}{s(s+k)(s-k)} \tag{22}$$

Inverting to the time domain gives

$$U_{(t)} = \frac{P_D}{k} + \left[\frac{k(U_o + D_o) - (P_U + P_D)}{2k}\right]e^{-kt} + \left[\frac{k(U_o - D_o) + (P_u - P_D)}{2k}\right]e^{kt} \tag{23}$$

If destroyers are added at a rate $P_D = 15$, the number of U-boats goes to zero at $t_F = 16.5$ weeks, and the number of remaining destroyers is $D = 71.8$. If, however, destroyers are added at a slightly reduced rate $P_D = 14$, the number of destroyers goes to zero at $t_F = 11.2$ weeks, and the number of remaining U-boats is $U = 81.2$. Figure 3 shows the dynamic changes in the number of vessels in each fleet for the two cases. Thus, Admiral Steadman's claim that 15 ships are needed per week is correct, and the Secretary of the Navy's claim of 5 ships per week is a gross underestimate.

## FINAL EXAMINATION PROBLEM

The final sequential problem moves into future star wars. The battle is between Captain Kirk's fleet of starships and the evil Klingon fleet in the year 2200. Now there are two types of starships: some have better defensive shields and some have more firepower than the Klingon ships. The mathematical model now has three ordinary differential equations with different coefficients in the destruction-rate term.

### ■ *Problem Statement*

*Captain James Kirk is in command of a fleet of 16 starships of the Enterprise class. A Klingon fleet of 20 ships has been spotted approaching. The legendary Lt. Spock has recently retired, so Captain Kirk turns to his new intelligence officer, Lt. Steadman (Lehigh Class of 2196 in chemical engineering) for a prediction of the outcome of the upcoming battle. Steadman has been working with the new engineering officers in the fleet, Lt. Moquin and Lt. Walsh, who have replaced the retired Lt. Scott. These innovative officers have been able to increase the firepower of half of the vessels in Kirk's fleet by a factor of two over the firepower of the Klingon vessels, which all have the same firepower. The firepower of the rest of Kirk's fleet is on a par with that of the Klingons. But these officers have also been able to improve the defensive shields on the second half of the fleet. The more effective shields reduce by 50% the destruction rate of these vessels by the Klingon firepower.*

*Thus, there are two classes of starships: eight vessels are Class $E_1$, with increased firepower, and eight vessels are Class $E_2$, with improved defensive shields. Assume that half of the Klingon fleet is firing at each class at any point in time.*

*Calculate who wins the battle and how many vessels of each type survive.*

### ■ *Problem Solution*

The dynamic model of the system is

$$\frac{dE_1}{dt} = -k\left(\frac{K}{2}\right) \tag{24}$$

$$\frac{dE_2}{dt} = -\left(\frac{k}{2}\right)\left(\frac{K}{2}\right) \tag{25}$$

$$\frac{dK}{dt} = -kE_2 - 2kE_1 \tag{26}$$

with the initial conditions $E_{1(t=0)} = 8$, $E_{2(t=0)} = 8$, and $K_{(t=0)} = 20$. Laplace transforming and combining give

$$K_{(s)} = \frac{20s - 24k}{s^2 - \frac{5}{4}k^2} = \frac{20.73}{s + 1.118k} - \frac{0.7335}{s - 1.118k} \tag{27}$$

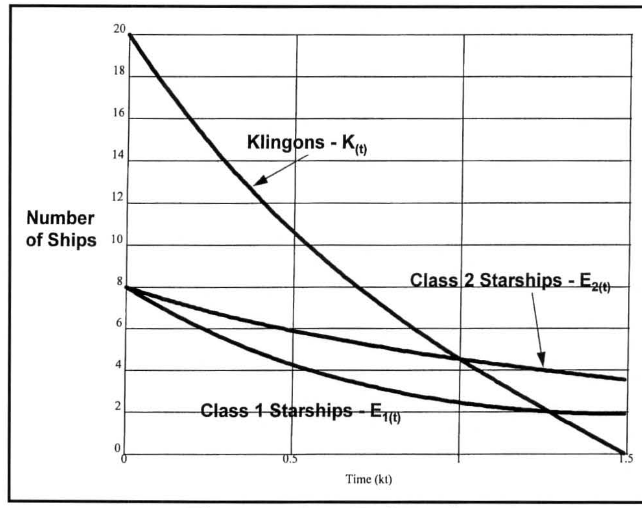



***Figure 4.*** *Star Trek Battle.*

$$E_{1(s)} = \frac{8 - kK_{(s)}/2}{s} \tag{28}$$

$$E_{2(s)} = \frac{8 - kK_{(s)}/4}{s} \tag{29}$$

Inverting to the time domain gives

$$K_{(t)} = 20.73e^{-1.118kt} - 0.7335e^{1.118kt} \tag{30}$$

$$E_{1(t)} = 9.271e^{-1.118kt} + 0.328e^{1.118kt} - 1.599 \tag{31}$$

$$E_{2(t)} = 4.636e^{-1.118kt} - 0.164e^{1.118kt} + 3.528 \tag{32}$$

The end of the battle occurs when $K_{(t_F)} = 0$. Solving the first Eq. 30 yields $t_F = 1.494/k$. The number of surviving starships is $E_{1(t=1.494/k)} = 1.889$ and $E_{2(t=1.494/k)} = 3.528$. So it is better to be on a ship with better shields than on a ship with more firepower in this matchup. Figure 4 shows the dynamic changes in the number of vessels in each fleet.

## CONCLUSION

This sequential non-chemical engineering problem illustrates the basic ideas of the teaching methods proposed in this paper. Students respond when you show them how they can apply the fundamental principles they are learning about chemical engineering processes to many other real-life situations. "Variations-on-a-theme" problems help students learn the basic principles of dynamic modeling in a variety of situations. They learn how to think and how to derive models instead of trying to find a formula in a book.

## REFERENCES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2. Luyben and Luyben, *Essentials of Process Control,* McGraw Hill Book Company, New York, NY, Problems 7.23, 7.24, 7.28, and 7.29 (1997) ❒